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Asymptotic Approximation by Bernstein–Durrmeyer Operators and their Derivatives

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Abstract

The concern of this paper is the study of local approximation properties of the Bernstein–Durrmeyer operators M_n*. We derive the complete asymptotic expansion of the operators* M_n *and their derivatives as* n *tends to infinity. It turns out that the appropriate representation is a series of reciprocal factorials. All coefficients are calculated explicitly in a very concise form. Our main theorem contains several earlier partial results as special cases. Moreover, it may be useful for further investigations on Bernstein–Durrmeyer operators. Finally, we obtain a Voronovskaja–type formula for the simultaneous approximation by linear combinations of the* M_n*.*

1 Introduction

The Bernstein–Durrmeyer operators M_n introduced by Durrmeyer [14] and, independently, by Lupas [21, p. 68] associate with each function f integrable on $I = [0, 1]$ the polynomial $M_n f$ defined by

$$(M_n f)(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)\, dt \qquad (x \in I),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad (0 \le k \le n).$$

They result from the classical Bernstein operators $(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(\frac{k}{n})$ by replacing the discrete values $f(\frac{k}{n})$ by the integral $\int_0^1 p_{n,k}(t) f(t)\, dt$ in order to approximate L_p functions $(1 \le p \le \infty)$.

The operators M_n were studied by Derriennic [11] and several other authors. It was shown that M_n are positive contractions in $L_p(I)$ and self adjoint on $L_2(I)$. Moreover, the operators commute, that is, $M_n M_k f = M_k M_n f$ for all $n, k \in \mathbb{N}$. Among other things Derriennic [11, Théorème II.5] (see also [16, Lemma 1.1] and [10, (i), p. 59]) found the Voronovskaja–type formula

$$\lim_{n\to\infty} n\left((M_n f)(x) - f(x)\right) = (1-2x) f'(x) + x(1-x) f''(x) \tag{1}$$

for all bounded integrable functions f on I admitting a derivative of second order at x $(x \in I)$. The first result of this type was given by Voronovskaja [24] for the classical Bernstein polynomials and then generalized by Bernstein [9].

Our Theorem 1 contains (as special case $r = 0$) the complete asymptotic expansion for the Bernstein–Durrmeyer operators by means of a series of reciprocal factorials, i.e.,

$$(M_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{1}{(n+2)^{\overline{k}}} \left(\frac{(x(1-x))^k f^{(k)}(x)}{k!} \right)^{(k)} \qquad (n \to \infty), \tag{2}$$

provided $f \in L_\infty(I)$ and f possesses derivatives of sufficiently high order at x $(x \in I)$. Throughout the paper $n^{\overline{k}}$ resp. $n^{\underline{k}}$ denotes the rising factorial $n^{\overline{k}} = n(n+1)\cdots(n+k-1)$, $n^{\overline{0}} = 1$ resp. falling factorial $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$, $n^{\underline{0}} = 1$. Formula (2) means that, for all $q \in \mathbb{N}$,

$$(M_n f)(x) = f(x) + \sum_{k=1}^{q} \frac{1}{(n+2)^{\overline{k}}} \left(\frac{(x(1-x))^k f^{(k)}(x)}{k!} \right)^{(k)} + o(n^{-q})$$

as $n \to \infty$. The above–mentioned Voronovskaja–type result (1) is the special case $q = 1$.

It is amazing that to our best knowledge such a nice result does not appear in the literature up to the present. In particular, the special case for polynomial f may be useful for further investigations on Bernstein–Durrmeyer operators.

We remark that in [1, 3, 2, 4, 7] the author gave analogous results for the operators of Meyer–König and Zeller, for the operators of Bleimann, Butzer and Hahn, the Bernstein–Kantorovich operators, and the operators of K. Balázs and Szabados, respectively. Asymptotic expansions of bivariate operators can be found in [5, 6].

Concerning simultaneous approximation already Derriennic [11, Théorème II.6] showed that

$$\lim_{n\to\infty} \left(\frac{d}{dx}\right)^r (M_n f)(x) = f^{(r)}(x)$$

for all $f \in L_\infty(I)$ admitting a derivative of order r at the point $x \in I$. Agrawal and Kasana [8] proved the generalization

$$\lim_{n\to\infty} n \left(\frac{(n+r+1)!\,(n-r)!}{(n+1)!\,n!} (M_n^{(r)} f)(x) - f^{(r)}(x) \right)$$

$$= (r+1)(1-2x) f^{(r+1)}(x) + x(1-x) f^{(r+2)}(x), \tag{3}$$

if f admits, in addition, a derivative of order $r+2$ at x.

Using an auxiliary operator introduced by Heilmann and Müller [17] we prove in Theorem 1 that the complete asymptotic expansion for the differentiated operators $(M_n^{(r)} f)$ can be obtained by differentiating r times the terms in expansion (2), i.e.,

$$(M_n^{(r)} f)(x) \sim f^{(r)}(x) + \sum_{k=1}^{\infty} \frac{1}{(n+2)^{\overline{k}}} \left(\frac{(x(1-x))^k f^{(k)}(x)}{k!} \right)^{(r+k)} \tag{4}$$

as $n \to \infty$, provided $f^{(r)} \in L_\infty(I)$ and f possesses derivatives of sufficiently high order at x $(x \in I)$.

The Voronovskaja–type formula

$$\lim_{n\to\infty} n \left((M_n f)(x) - f(x) \right)^{(r)} = \left(x(1-x) f'(x) \right)^{(r+1)}$$

contained in Eq. (4) is due to Heilmann [17, Theorem 8].

Note that our Formula (4) immediately implies the result (3) of Agrawal and Kasana since

$$\frac{(n+r+1)!\,(n-r)!}{(n+1)!\,n!} = 1 + \frac{2}{n} \binom{r+1}{2} + O(n^{-2}) \qquad (n \to \infty).$$

We close the manuscript with the complete asymptotic expansion for the simultaneous approximation by linear combinations

$$(O_{n,\ell} f)(x) = \sum_{i=0}^{\ell-1} \alpha_i(n) \left(M_{n_i} f \right)(x)$$

of the Bernstein–Durrmeyer operators M_n used by Ditzian and Ivanov [13] (see also Heilmann [18, pp. 87ff]).

2 The main result

For $r, q = 0, 1, 2, \ldots$ and $x \in I$, let $K[r, q; x]$ be the class of all functions $f \in L_\infty^r(I)$ which are $r+q$ times differentiable at x. Throughout the paper put, as usual, $\varphi(x) = \sqrt{x(1-x)}$. As main result we formulate the following theorem.

Theorem 1 *Let $r \in \mathbb{N}_0$, $q \in \mathbb{N}$ and $x \in I$. Then, the Bernstein–Durrmeyer operators M_n satisfy, for $f \in K[r, 2q; x]$, the asymptotic relation*

$$(M_n^{(r)} f)(x) = f^{(r)}(x) + \sum_{k=1}^{q} \frac{1}{(n+2)^{\overline{k}}} \left(\frac{\varphi^{2k}(x) f^{(k)}(x)}{k!} \right)^{(r+k)} + o(n^{-q}) \qquad (5)$$

as $n \to \infty$, where $\varphi(x) = \sqrt{x(1-x)}$.

Remark 1 *For $f \in \bigcap_{q=1}^{\infty} K[r, q; x]$, the Bernstein–Durrmeyer operators M_n possess the complete asymptotic expansion*

$$(M_n^{(r)} f)(x) \sim f^{(r)}(x) + \sum_{k=1}^{\infty} \frac{1}{(n+2)^{\overline{k}}} \left(\frac{\varphi^{2k}(x) f^{(k)}(x)}{k!} \right)^{(r+k)}$$

as $n \to \infty$.

For the convenience of the reader we calculate the explicit form of the asymptotic expansion (5) for $q = 2$.

Corollary 2 *Let $r \in \mathbb{N}_0$ and $x \in I$. Then, the Bernstein–Durrmeyer operators M_n satisfy, for $f \in K[r, 4; x]$, the asymptotic relation*

$$\begin{aligned}
&(M_n^{(r)} f)(x) = f^{(r)}(x) \\
&\quad + \frac{1}{n+2} \Big(x(1-x) f^{(r+2)}(x) + (r+1)(1-2x) f^{(r+1)}(x) - (r^2+r) f^{(r)}(x) \Big) \\
&\quad + \frac{1}{(n+2)(n+3)} \Big((x^4 - 2x^3 + x^2) f^{(r+4)}(x) \\
&\quad + 2(r+2)(2x^3 - 3x^2 + x) f^{(r+3)}(x) + (r+2)(r+1)(6x^2 - 6x + 1) f^{(r+2)}(x) \\
&\quad -2(r+2)(r^2+r) f^{(r+1)}(x) - (r+2)(r^3 - r) f^{(r)}(x) \Big) + o(n^{-2})
\end{aligned}$$

as $n \to \infty$.

3 Linear combinations of M_n–operators

In this section we give an application of Theorem 1. We study the local simultaneous approximation by linear combinations of the Bernstein–Durrmeyer operators M_n.

As in [13, Eq. (5.1), (5.3)] we define, for fixed $\ell \in \mathbb{N}$,

$$(O_{n,\ell} f)(x) = \sum_{i=0}^{\ell-1} \alpha_i(n) \, (M_{n_i} f)(x) \,, \tag{6}$$

where

$$n = n_0 < n_1 < \cdots < n_{\ell-1} \leq An \tag{7}$$

with a constant A independent of n. In the following we put

$$\alpha_i(n) = (n_i + 2)^{\overline{\ell-1}} \prod_{\substack{j=0 \\ j \neq i}}^{\ell-1} (n_i - n_j)^{-1}. \tag{8}$$

In the case $\ell = 1$ the $O_{n,\ell}$ reduce to the operators M_n if in definition (8) the coefficient is interpreted to be $\alpha_i(n) = 1$.

Ditzian and Ivanov [13] as well as Heilmann [18] proposed the further condition

$$\sum_{i=0}^{\ell-1} |\alpha_i(n)| \leq B \tag{9}$$

with a constant B independent of n. We do not require (9) here. We point out that the choice (8) guarantees that condition (9) is valid, if we assume, in addition, that $n_{i+1} \geq \gamma n_i$ $(i = 0, \ldots, \ell - 1)$ with some constant $\gamma > 1$.

Theorem 3 *Let $\ell, q \in \mathbb{N}$, $r \in \mathbb{N}_0$, and $x \in I$. Then, the linear combinations $O_{n,\ell}$ as defined in Eqs. (6)–(8) satisfy, for $f \in K[r, 2(q+\ell); x]$, the asymptotic relation*

$$(O_{n,\ell}^{(r)} f)(x) = f^{(r)}(x) + \sum_{k=0}^{q} S(k, \ell; n_0, \ldots, n_{\ell-1}) \left(\frac{\varphi^{2(k+\ell)}(x) f^{(k+\ell)}(x)}{(k+\ell)!} \right)^{(r+k+\ell)} + o(n^{-(q+\ell)}) \tag{10}$$

as $n \to \infty$, where $\varphi(x) = \sqrt{x(1-x)}$ and

$$S(k, \ell; n_0, \ldots, n_{\ell-1}) = \frac{(-1)^{\ell+1}}{k!} \sum_{\nu=0}^{k} (-1)^{\nu} \binom{k}{\nu} \prod_{j=0}^{\ell-1} (n_j + \ell + 1 + \nu)^{-1}. \tag{11}$$

Moreover, we have

$$S(k, \ell; n_0, \ldots, n_{\ell-1}) = O(n^{-(k+\ell)}) \qquad (n \to \infty). \tag{12}$$

Remark 2 *Eq. (10) reveals the well–known fact that the operators $O_{n,\ell}$ preserve all polynomials of degree at most $\ell - 1$.*

Remark 3 *For $q = 0$, Theorem 2 yields the Voronovskaja–type formula*

$$\lim_{n\to\infty}\left[\prod_{j=0}^{\ell-1}(n_j+\ell+1)\right]\left(\left(O_{n,\ell}f\right)(x)-f(x)\right)^{(r)}$$

$$= (-1)^{\ell+1}\left(\frac{\varphi^{2\ell}(x)f^{(\ell)}(x)}{\ell!}\right)^{(r+\ell)}. \quad (13)$$

The special case $r = 0$ of Eq. (13) is due to Heilmann [18, Satz 8.4].

Remark 4 *For $f \in \bigcap_{q=1}^{\infty} K[r,q;x]$, we have the complete asymptotic expansion*

$$(O_{n,\ell}^{(r)}f)(x) \sim f^{(r)}(x)$$
$$+(-1)^{\ell+1}\sum_{k=\ell}^{\infty} S(k-\ell,\ell;n_0,\ldots,n_{\ell-1})\left(\frac{\varphi^{2k}(x)f^{(k)}(x)}{k!}\right)^{(r+k)}$$

as $n\to\infty$ with $S(k,\ell;n_0,\ldots,n_{\ell-1})$ as defined in Eq. (10).

Remark 5 *We remark that Eq. (12) follows easily if condition (9) is assumed (see [18, Lemma 2.3]). We prove (12) without making use of (9).*

4 Auxiliary results

The starting–point is the calculation of the moments $\left(M_n^{(r)}e_m\right)(x)$ for the differentiated Bernstein–Durrmeyer operators, where $e_m(x) = x^m$ $(m = 0, 1, 2, \ldots)$.

Proposition 4 *For $m, r = 0, 1, 2, \ldots$, the moments for the differentiated Bernstein–Durrmeyer operators possess the representation*

$$(M_n^{(r)}e_m)(x) = \sum_{k=0}^{m}\frac{1}{(n+2)^{\overline{k}}}\binom{m}{k}\left(x^m(1-x)^k\right)^{(r+k)} \qquad (n\in\mathbb{N}). \quad (14)$$

Remark 6 *Formula (14) yields for each polynomial P the representation*

$$(M_n^{(r)}P)(x) = \sum_{k=0}^{\infty}\frac{1}{(n+2)^{\overline{k}}}\left(\frac{(x(1-x))^k P^{(k)}(x)}{k!}\right)^{(r+k)} \qquad (n\in\mathbb{N}), \quad (15)$$

i.e., Eq. (4) is valid for polynomial f.

Note that the sum in Eq. (15), actually, is finite, since all terms for $k >$ degree P vanish. Furthermore, $M_n^{(r)}P = 0$, if $r >$ degree P. In particular, this shows the well–known fact that (M_nP) is a polynomial with degree $M_nP \leq$ degree P.

For $p \geq 1$ and $r \in \mathbb{N}$, let $L_p^r(I)$ be the class of all functions f with $f^{(r-1)}$ absolutely continuous on I and $f^{(r)} \in L_p(I)$. For $r = 0$, put $L_p^0(I) = L_p(I)$.

As in [17, 15] the operators

$$(M_{n,r}f)(x) = \frac{(n+1)!\ n!}{(n+r)!\ (n-r)!} \sum_{k=0}^{n-r} p_{n-r,k}(x) \int_0^1 p_{n+r,k+r}(t) f(t)\ dt$$

$$(r = 0, 1, 2, \ldots; n \geq r)$$

play an important role in the following. Integrating by parts r times we obtain, for $f \in L_p^r(I)$, the identity

$$M_n^{(r)} f = M_{n,r} f^{(r)}$$

(see [11, proof of Théorème II.8]) which is of use in the proofs.

We proceed in deriving the central moments for the operators $M_{n,r}$. For each fixed $x \in \mathbb{R}$, put $\psi_x(t) = t - x$.

Proposition 5 *For $r, s = 0, 1, 2, \ldots$ and $n \geq r$, we have*

$$(M_{n,r}\psi_x^s)(x) = s! \sum_{k=\lfloor (s+1)/2 \rfloor}^{r+s} \frac{1}{k!\ (n+2)^{\overline{k}}} \binom{r+k}{2k-s} \left(\frac{d}{dx}\right)^{2k-s} \varphi^{2k}(x).$$

In order to derive as our main result the complete asymptotic expansion of the Bernstein–Durrmeyer operators we use a general approximation theorem for positive linear operators due to Sikkema [22, Theorem 3] (cf. [23, Theorems 1 and 2]).

Theorem 6 *For $q \in \mathbb{N}$ and fixed $x \in I$, let $A_n : L_\infty(I) \to C(I)$ be a sequence of positive linear operators with the property*

$$(A_n \psi_x^s)(x) = O(n^{-\lfloor (s+1)/2 \rfloor}) \quad (n \to \infty) \quad (s = 0, 1, \ldots, 2q+2).$$

Then, we have for each $f \in L_\infty(I)$ which is $2q$ times differentiable at x the asymptotic relation

$$(A_n f)(x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} (A_n \psi_x^s)(x) + o(n^{-q}) \qquad (n \to \infty). \tag{16}$$

If, in addition, $f^{(2q+2)}(x)$ exists, the term $o(n^{-q})$ in (16) can be replaced by $O(n^{-(q+1)})$.

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