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Wilfried Hausmann

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On The Two Envelope Paradox

Wilfried Hausmann

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Adresse: Prof. Dr. Wilfried Hausmann, Fachbereich MND, Fachhochschule Giessen-Friedberg, Wilhelm Leuschner Str. 13, 61169 Friedberg/Hessen, e-Mail: Wilfried.Hausmann@mnd.fh-friedberg.de, Fax: 06031-604-185

Zusammenfassung: Das Zwei-Umschläge-Paradoxon ist ein schönes Beispiel dafür, wie ein subjektivistischer Wahrscheinlichkeitsbegriff in Kombination mit a-posteriori-Daten zu irreführenden Ergebnissen führen kann. Aber damit ist die Geschichte noch nicht zu Ende. In dieser Arbeit wird gezeigt, dass man mit Hilfe einer Art gemischten Strategie (im spieltheoretischen Sinn) sehr wohl Nutzen daraus ziehen kann, zu wissen, welcher Betrag sich in dem ersten Umschlag befindet. Damit ist der erste Umschlag als Preis für den zweiten eindeutig zu billig und es stellt sich die Frage nach einem fairen Preis. Auch wenn ein solcher im strengen Sinne nicht existiert, so lautet schließlich das Ergebnis, dass ein Aufschlag von 25% den "richtigen" Preis ergibt und das ist (Ironie des Schicksals oder tieferer Sinn?) genau der Preis, den die falsche subjektivistische Argumentation auf Anhieb liefert.

Summary. In the two envelope problem, unconditionally switching to the second envelope does not improve chances compared to the pure strategy of keeping the first one. But the amount of money to be won can be notably increased by a 'mixed' strategy.

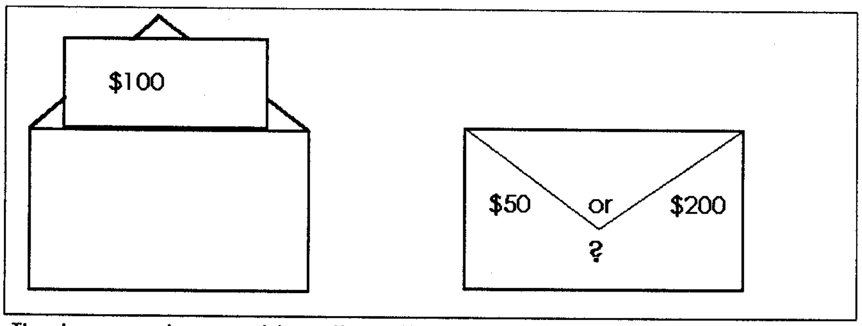
Schlüsselwörter: Spieltheorie, gemischte Strategien, subjektivistischer/objektivistischer Wahrscheinlichkeitsbegriff, faire Preise, Unterhaltungsmathematik

Key words: Game theory, mixed strategies, subjective/objective probabilities, fair prices, mathematical entertainment (recreational mathematics)

1. Introduction

By the 'Two Envelope Paradox' we understand the following situation (cf. [1])

> *Someone is asked to select one of two envelopes and is told that one contains twice as much money as the other. He finds $100 in the chosen envelope. If he is allowed to switch to the other one should he do so?*



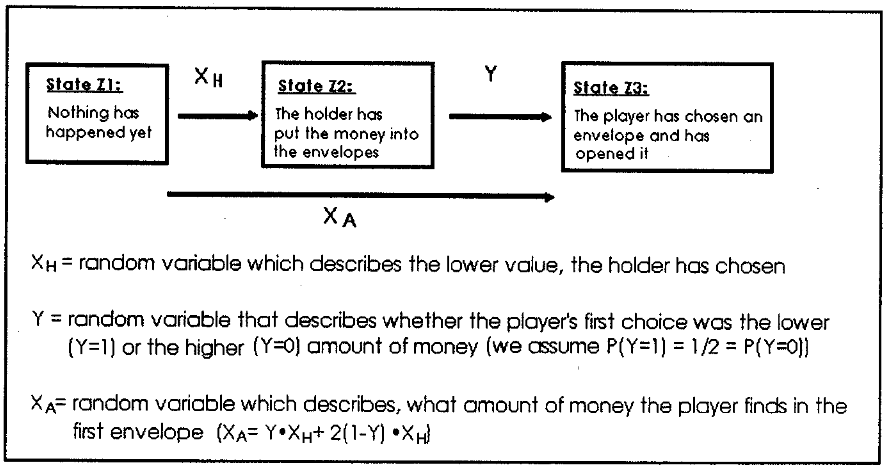
The two envelope problem: Keep the first envelope or switch to the second? The author's advice if you don't know what to do: Ask a man on the street!

Naive subjective argumentation suggests that it is preferable to switch: By even odds, the candidate will find either \$50 or \$200 in the other envelope. That makes an average (= expected value) of \$125. Since this type of argumentation works for every possible value in the first envelope, we are led to the conclusion that switching to the other one generally improves the expected amount of money by 25%, which, of course, is absurd.

Now, objective argumentation puts things back to where they are expected: From the moment that the two envelopes are filled, the expected value of switching after having found \$100 in the first envelope is objectively either -\$50(=\$50-\$100) or \$100(=\$200 - \$100), and there is no sensible way of averaging between the two possibilities. If one wishes to compare the two possibilities, he has to extend the probability model and include the process of filling the envelopes by the 'holder'. Thus, the total situation can be modelled by states and random variables describing how one state leads to the next one, as shown in the figure on page 3.

In [1] E. Linzer explains that in this model the two strategies, 'unconditionally switching' and 'never switching', on average will lead to the same amount of money. Moreover, to judge chances in the particular situation of having found \$100 in the first envelope, one needs information about the distribution of X_H, namely the probabilities $P(X_H = 50)$ and $P(X_H = 100)$, which, in general, is not available (for simplicity we assume throughout this article, that X_H is a discrete random variable. With the exception of section 4.1 we also assume always, that X_H has finite expected value).

While everything that is written in [1] surely is correct, it is of poor practical use. Even if one does know the approximate shape of the distribution of X_H, this does not help very much since one needs to have information about the very special values $P(X_H = 50)$ and $P(X_H = 100)$. And couldn't it, for instance, be true that $P(80 < X_H \leq 120)$ is large, but $P(99 < X_H \leq 101)$ is near or equal to zero? More



helpful advice can be given, for instance

Ask someone who is not concerned with the matter, to tell you an arbitrary positive number, and decide as follows:
If the number you are told is less than 100, then keep the first envelope, otherwise take the second one

is a good advice to the candidate, which surprisingly does improve his chances whatever the distribution of X_H may be.

Why the answer by chance of someone who does not have the slightest idea what it is all about does help the candidate will be explained in Chapter 3, where also some more systematic strategies of the candidate are introduced. In the Chapters 2 and 4 we tackle the problem of determining a fair price of the second envelope, when the content of the first (=$100 in the original formulation of the problem) is known. It turns out, that $100 is a fair price if the candidate has to buy the second envelope. This follows immediately from the well known solution of the paradox. But from the results of Chapter 3 it becomes clear that a price of $100 no longer remains fair if there is a choice to buy the second envelope or not. In fact, in this case there are good reasons to call $125 a fair price, which, again, is quite surprising, as this is the price one is led to by the (wrong) subjective argumentation.

To start the explanation, let us first reformulate and generalize the problem:

> *Two people agree to play the following game: One of them (the* **holder** *or* **vendor***) puts money into two envelopes so that one envelope contains twice as much as the other. The second person (the* **player** *or* **candidate***) then chooses one of the envelopes, opens it and finds some amount of money (x_1) in it. The player now can buy the second envelope at a price*

depending on the value of x_1. Envelope one and the money in it are returned to the vendor.

2. Fair prices (part 1)

We are looking for fair conditions in this game, i.e. conditions such that none of the parties has an advantage from the beginning. In this chapter, we will study the simpler case where the player is obligated to buy the second envelope.

We assume that the price of the second envelope depends on x_1, the value of the first and that the method for deriving this price from x_1 is fixed before the game starts and may be expressed by a function G. We call such a function a **price agreement** and say that a price agreement is **fair** iff the expected value of $G(X_A)$ equals the expected value of X_B. Here X_A and X_B denote random variables which describe the amounts in envelope one and two, respectively.

There are many fair price agreements, but two of them are of special interest: first, there is G_1 defined by

$$G_1(x_1) := x_1$$

We shall call G_1 the **canonical price agreement**. It means, that the player has to pay the sum of envelope one to buy envelope two. This, of course, is equivalent to switching to the second envelope in the original formulation of the problem.

The previously mentioned fact, that in the original problem the strategies 'always switching' and 'never switching' give the same result, means that G_1 is fair. It is remarkable that the fairness of this price agreement does not depend upon the fact that one envelope contains twice as much money as the other does. It would still be true if the two envelopes contained any amount of money with no further conditions imposed. The values could even have been chosen independently by two different persons. This seems very strange at first glance as it appears to mean that we make a reasonable guess about a random variable based on a second, independent variable. However, although independent from each other, there is a link between the two random variables. This link is given by the fact that in state Z_2, since the odds are even, the player might as well have taken the other envelope.

The following two remarkable properties of G_1 should also be observed:

- G_1 does not depend on X_H, i.e. it is fair, whatever the distribution of X_H may be.
- The conditional expected value $E(G_1(X_A)|X_H = x)$ equals $E(X_B|X_H = x)$ for all values of x. This means that G_1 is still fair when state Z_2 is reached. It is the only general price agreement with this property.

This second property is likely to soothe a player who suspects that the holder is trying to double-cross him. In contrast, price agreement G_2, which we will now define, would surely not be accepted by such a player. Let $P_H(x) := P(X_H = x)$ and define price agreement G_2 by

$$G_2(x_1) \;:= x_1 \cdot \frac{2P_H(x_1) + 1/2\,P_H(x_1/2)}{P_H(x_1) +\, P_H(x_1/2)}$$

For reasons which will soon be clear, we shall call G_2 the **theoretical price agreement**. An easy calculation shows that for each value x_1, the player may find in the first envelope, the equation

$$E(G_2(X_A)|X_A = x_1) = E(X_B|X_A = x_1) \tag{1}$$

holds. Hence the theoretical price agreement is fair. However, it has several apparent disadvantages:

- In general, when state Z_2 is reached one of the competitors already has an advantage.
- Because of its extreme dependence on the random variable X_H which can be manipulated by the vendor, it would in reality be hard to make the player accept price agreement G_2.

There is still one important advantage of the theoretical price agreement: Because of (1) G_2 stays fair even if the player in state Z_2 is allowed to withdraw from the purchase of envelope two. As will be shown in the next chapter, the canonical price agreement G_1 does not have this property.

3. Promising strategies for the player

It will be assumed throughout this chapter that price agreement G_1 is agreed upon and that the player is free to buy or not to buy. Thus we will consider the exact situation of the original problem.

3.1. The C_{max}-strategy. As previously shown, the strategies 'always buying (or switching) and 'never buying' have the same expected value: 0. In order to be more successful, the player has to do something different. If he knew the random variable X_H, it would be advantageous to switch whenever the canonical price is less than the theoretical price and otherwise to keep. But since he does not know X_H, what can he do?

Well, there is a toe-hold: Whatever the distribution of X_H may be, the distribution of $2X_H$ is placed further to the right on the real axis than the distribution of X_H. And this toe-hold is sufficient:

Definition 1. *($C_{\max}$-strategy) Choose a positive number $C_{\max}$ at random and decide in the following way: If x_1 is less than or equal to $C_{\max}$, then buy the second envelope, otherwise do not.*

If the random method of choosing $C_{\max}$ is appropriate, this mixed strategy will beat the pure strategies of 'always buying' and 'never buying':

Theorem 2. *If $C_{\max}$ is chosen in such a way that for each interval I of positive length the probability $P(C_{max} \in I)$ is greater than zero, then the expected value of the strategy is positive. This holds for all possible values \$$x$ and \$$2x$ in the two envelopes and consequently is true for all distributions of X_H.*

Proof. We look at the situation from state Z_2. Therefore, the lower sum (\$$x$) and the higher sum (\$$2x$) have already been chosen. There are three different cases to consider:
i) $C_{\max} \geq 2x$. In this case, the $C_{\max}$-strategy coincides withe the pure strategy of always buying which has expected value zero.
ii) $x > C_{\max}$. In this situation, the player will not buy regardless of which envelope he takes first. Again, his balance on average will be zero.
iii) $2x > C_{\max} \geq x$. In this case, the player will act in an optimal way: if $x_1 = x$, which happens with probability 1/2, he will buy the second envelope at the price of \$$x$ and thus gain \$$x$. On the other hand, if $x_1 = 2x$, he does not buy and thus avoids a loss of \$$x$.
Case iii) happens with probability $P\left(C_{max} \in [x, 2x)\right)$, which by assumption is greater than zero. Therefore the conditional expected value of the strategy under the assumption $X_H = x$ equals $\frac{1}{2}x \cdot P\left(C_{max} \in [x, 2x)\right)$, which implies that the expected value (=expected profit) in total is positive.

Remark 1. *i) Of course, there exist ways of choosing C_{max}, such that the condition of the theorem is fulfilled. For instance, the player might choose C_{max} as realisation of a random variable with distribution function $F(x) = 1 - 1/(1+x)$ $(x > 0)$.*
ii) It should be emphazised that the theorem holds whatever the distribution of X_H may be. In any event, the success of the C_{max}-strategy is not completely independent of X_H. In general, it is true that the expected value of the strategy is greater than zero, but its precise value depends upon X_H and how the distribution of X_H and the way of choosing C_{max} fit together. Consequently, one might suspect that the advantage of the strategy is merely of a theoretical nature, but this is not true as we shall see below. With only very few assumptions about X_H, the strategy can gain a considerable amount of money.
iii) It is an astonishing fact that the theorem does not depend on the fact that the values $v(= x)$ and $w(= 2x)$ in the two envelopes are linked by the relation $w = 2v$.

The argument works whenever the two sums are different. So the theorem remains true even if v and w are chosen completely independently. This is due to the fact that under the assumption of the theorem, for any positive numbers $v < w$ the probability $P(C_{max} > v)$ is greater than $P(C_{max} > w)$.

The C_{max}-strategy can be reformulated and slightly generalized in the following way: For a nonnegative number x_1, let $P_A(x_1) := P(C_{max} > x_1)$. In other words,

$$P_A(x_1) = P(\text{buying}|\ X_A = x_1)$$

is the conditional probability that this particular C_{max}-strategy will lead to the decision to buy when the first envelope contains \$$x_1$. Then P_A is a monotone decreasing function with (most likely) $P_A(0) = 1$ and $P_A(x_1) \to 0$ as x_1 tends to infinity. If the C_{max}-strategy fulfills the assumption of the theorem, P_A is strictly monotone. Conversely, one might start with a positive strictly monotone decreasing function P_A with $P_A(0) = 1$ and derive a C_{max}-strategy from it by applying the inverse function of P_A to a standard $U(0,1)$-random number (if P_A does not meet (0,1) surjectively, this definition has to be modified accordingly)

Actually, the function P_A need not be strictly monotone decreasing everywhere to guarantee the evidence of the theorem. It is sufficient to have P_A strictly decreasing on all sequences of the type

$$..., 2^{-k}a, ..., 2^{-1}a, a, 2a, ..., 2^k a, ...$$

where a is a positive real number. So one might, for instance, choose P_A as a constant on intervals of the type $\left[2^k, 2^{k+1}\right)$. If there is a choice, it is preferable to define $P_A(x_1)$ to be as large als possible as this inreases the performance-rate (=probability of an accomplished purchase).

We shall now introduce several strategies which require some assumptions about the distribution of X_H such as lower or upper bounds.

3.2. A strategy with predetermined expected success-rate (lower bound assumed). We now assume that there is a lower bound L for the distribution of X_H. If such a lower bound does not exist, the following still remains true for all cases where the actual value of X_H is not less than L (this remark is not meaningless as the player might wish to fix a lower bound so that he would buy unconditionally when viewing an amount of money smaller than this lower bound).

We are looking for a strategy S_1 that guarantees a predetermined success-rate p $(1 > p > \frac{1}{2})$. By this we mean that - whatever the distribution of X_H may be - the conditional probability $P(X_H = X_A|\, S_1 \text{ tells to buy})$ must be equal to p.

It must be emphazised that this does not guarantee that for all possible values x_1

$$P(\text{bargain}|\ X_A = x_1) = p \text{ or } P(\text{bargain}|\ X_A = x_1 \text{ and } S_1 \text{ tells to buy}) = p$$

holds, since the latter two conditions require further knowledge of X_H.

Again, the key to the strategy is obtained by viewing the problem from state Z_2: The envelopes contain \$$x$ and \$$2x$ respectively with $x \geq L$. We assume our strategy is given by a function P_A as described at the end of 3.1. Then the probability p_1 that the player will buy the second envelope is

$$p_1 = \frac{1}{2}P_A(x) + \frac{1}{2}P_A(2x)$$

and the probability of a bargain is

$$p_2 = \frac{1}{2}P_A(x).$$

Hence, the conditional probability p_3 of a bargain under the assumptions a) $X_H = x$ and b) the strategy tells to buy - equals $p_3 = P_A(x)/(P_A(x) + P_A(2x))$. So $p_3 = p$ leads to the condition

$$P_A(2x) = P_A(x)(1 - p)/p.$$

Now the strategy is almost clear, but before stating it we need some definitions: Let for $x \geq L$ $\mathbf{d_L(x)}$ denote the maximal natural number k such that $x/2^k$ is a priori possible as an amount in one of the envelopes. Of course, $[log_2(x/L)] \geq d_L(x)$ ($[\quad] =$ integral part), but $d_L(x)$ can be much smaller as it may, for instance, be reasonable to assume that the amounts of money in the envelopes are integral dollar-values. We call $d_L(x)$ the **L-depth** of x. Moreover we define $\mathbf{Prim}_L(\mathbf{x}) := x/2^{d_L(x)}$ and finally call x **L-primitive**, iff $x = Prim_L(x)$.

We may now describe strategy S_1. Let x_1, as usual, denote the value found in the first envelope.

Definition 3. ***Strategy*** $\mathbf{S_1}$ ***with parameter*** $\mathbf{p}$ ***(*** $\frac{1}{2} < \mathbf{p} < \mathbf{1}$***)***: *Choose a standard random number* $u \in (0, 1)$*. Then check whether*

$$\left(\frac{1-p}{p}\right)^{d_L(x_1)} > u$$

If this is true, buy the other envelope, otherwise do not.

Hence we have put $P_A(x_1) = \left(\frac{1-p}{p}\right)^{d_L(x_1)}$. What is the expected surplus of money? Let x be the lesser of the two amounts in the envelopes. Then

$$\begin{aligned} E(S_1|X_H = x) &= 1/2 \cdot \left(\tfrac{1-p}{p}\right)^{d_L(x)} \cdot x - 1/2 \cdot \left(\tfrac{1-p}{p}\right)^{d_L(x)+1} \cdot x \\ &= \left(\tfrac{1-p}{p}\right)^{d_L(x)} \cdot \tfrac{2p-1}{2p} \cdot x \end{aligned}$$

The case $p = 2/3$ is of special interest. Applying strategy S_1 with this parameter guarantees that for each accomplished purchase the chances of a bargain are 2:1. In this case, the formula for the expected surplus of earned money simplifies to

$$E(S_1|X_H = x) = \frac{1}{4} Prim_L(x)$$

and hence is constant on the sequence

$$Prim_L(x),\ 2Prim_L(x),\ 4Prim_L(x),\ ...$$

Notice also that $E(S_1) \geq L/4$. We will return to this fact later, but will now illustrate the numerical properties of strategy S_1 by an example.

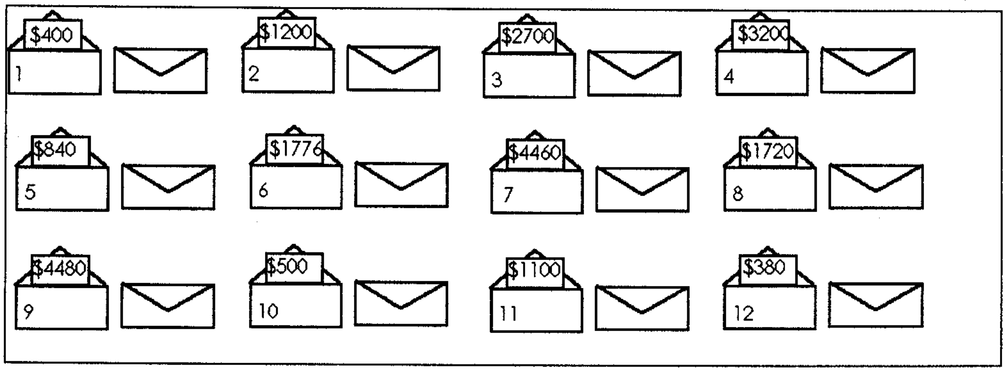
Results from computer simulation. For $p = 2/3$ we give the results of 10 series of 1000 games each. The agreed lower bound L was set to $L = 1$ (and only integral dollar values in the envelopes). The values of X_H were generated in the following way:
Series 1 to 5: $X_H = 2 + 2 \cdot \textit{Random}(5000)$
Series 6 to 10: $X_H = 2 + 5000 \cdot \textit{Random}(2) + 2 \cdot \textit{Random}(1250)$
Here $\textit{Random}(N)$ means a discrete random variable equally distributed on the set $\{0, ..., N-1\}$.

The meanings of the columns of the table are: **No** - Number of the series; **Low** - no. of occurences of $X_H = X_A$; **High** - No. of occurences of $X_H < X_A$; **Buy** - No of buys; **Gain** - No. of gains; **Loss** - number of losses; **Total offer** - total amount of money in the first envelopes; **Sum of buys** - total of money paid by the player to buy second envelopes; **Surplus** - balance from buys; **Percent** - = 100·surplus/(sum of buys)

No	Low	High	Buy	Gain	Loss	Total Offer	Sum of buys	Surplus	Percent
1	483	517	260	182	78	7,634,980	1,712,978	524,312	30.6
2	495	505	257	171	86	7,551,630	1,752,110	359,576	20.5
3	487	513	222	149	73	7,607,316	1,505,468	300,672	19.9
4	508	492	256	171	85	7,551,770	1,721,288	364,724	21.2
5	511	489	257	183	74	7,401,444	1,719,172	599,368	34.9
6	501	499	264	177	87	5,451,022	1,292,034	324,390	25.1
7	513	487	248	167	81	5,547,818	1,175,700	345,060	29.4
8	524	476	259	173	86	5,628,722	1,418,124	254,256	18.0
9	484	516	236	156	80	5,852,916	1,117,928	270,338	24.2
10	531	469	266	189	77	5,557,294	1,286,960	445,742	34.7

To be complete it should be mentioned here that in all cases, this strategy was much more successful than ‘always buying’.

In order not to overestimate the power of this strategy, it must be mentioned that distributions of X_H could have been chosen instead, such that the balance would have been much smaller. If, for instance, $Prim_L(x)$ equals 1 and the L-depth of x is high, the variance of a single game is very large whereas the expected surplus is only 25 cents.



In which cases should one switch to the second envelope?
The strategy S1 might tell you to take the second envelope in the cases 2, 5, 7, and 12. It might as well give a completely different advice (for instance 1, 3, 4, 10, 11). Nevertheless, in all simulations the strategy turned out to be very successful on the long run.

3.3. A strategy with proportional expected surplus (lower and upper bound assumed). Though not optimal with regard to the next chapter, the strategy S_2 that will now be introduced is very appropiate for practical games as it can be easily adapted to accomodate the player's private risk-preference: He may choose a lower bound L such that he would switch in any case if he finds less than $\$2L$ in the first envelope, and he may choose an upper limit U such that he will not take the risk of loosing $\$U/2$.

The following holds whenever X_H is greater than L and less than U. Therefore, the numbers L and U may be agreed upon by vendor and player or may just be derived from the player's own risk-preference.

Again we look at the situation from State Z_2. As usual, let x denote the smaller sum. If there is a positive number c such that

$$P_A(2x) = P_A(x) - c$$

holds, the player will have an expected surplus of

$$\frac{1}{2}P_A(x) \cdot x - \frac{1}{2}(P_A(x) - c) \cdot x = \frac{1}{2}c \cdot x.$$

In order to maximize c (which is desirable to the player), for x_1 between L and $2U$ let the **(L,U)-height $\mathbf{h}_{(L,U)}(\mathbf{x}_1)$** be defined as the smallest natural number k such that $2^k(Prim_L(x_1))$ is greater than or equal to U. Clearly $[1 + \log_2(U/L)] \geq h_{(L,U)}(x_1)$. Now we can complete the definition.

Definition 4. ***Strategy*** $\mathbf{S_2}$.: *Choose a standard random number* $u \in (0,1)$. *Then check whether*

$$u \geq d_L(x_1)/h_{(L,U)}(x_1).$$

If this is true, buy the other envelope otherwise do not.

Hence we have $P_A(x_1) = 1 - d_L(x_1)/h_{(L,U)}(x_1)$. Strategy S_2 thus satisfies

$$E(S_2|X_H = x) = \frac{1}{2}cx \tag{2}$$

with $c = 1/h_{(L,U)}(x_1) \geq 1/[1 + log_2(U/L)]$. Now assume that X_H is always between L and U and let c_1 be the minimum of the numbers c that occur in (2). Then

$$E(S_2) \geq \frac{1}{2}c_1 E(X_H) = c_1/3 \cdot E(X_A)$$

4. Fair prices (continued)

Fair conditions must be determined for the case in which the player is allowed not to buy. Since the player has a definite advantage if the canonical price agreement G_1 is used and since the theoretical agreement G_2 is unsuitable for practical use, one might ask if there are different ways to obtain evenly-balanced chances for both opponents.

4.1. Lower bound assumed. A first approach is to agree on a constant fee, not based on the amount of the first envelope, which the player must pay for the right to buy or not to buy the second envelope - no matter if he uses this right or not. Paying the fee can thus be thought of as buying a ‘call option’ for the second envelope. And indeed, by introducing such a fee, fair conditions can be reestablished, provided there is an agreement on a lower bound L:

Assume the holder chooses \$$x$ as the lower value. Then by applying strategy S_1 with $p = 2/3$ the player has an expected gain of $prim_L(x)/4$ (dollars). Thus, the player should have to pay at least this amount. On the other hand, suppose the holder chooses the distribution of X_H in the following way:

$$P(X_H = 2^k L) = 2^{-(k+1)} \text{ for } k = 0, 1, 2, ...$$

and $P(X_H = x) = 0$ for all other $x > L$. Then with probability $p = 1/4$ the player will find \$$L$ in the first envelope and, of course, will certainly buy envelope two and thus gain $\$L$ (minus the agreed upon fee). If the player finds a different value in the first envelope, he may choose to buy the second or not - both with the same expected value (when viewed from state Z_1), since in all these cases price agreement G_1 coincides with price agreement G_2 and, as pointed out above, when price agreement G_2 is used the player cannot take advantage (in the sense of positive expected value) from the free choice to buy or not (cf. (1)). Summing up, the player's expected gain from buying cannot exceed $\frac{1}{4}L$ if the vendor chooses X_H this way. Thus it seems reasonable to set the fee to $\$L/4$.

Let us introduce some additional notation: By a **strategy of the player** we mean the selection of a function P_A as in chapter 3 and by a **strategy of the holder** we mean the selection of a distribution of X_H. A price Q (for an option to buy the second envelope at the amount of the first) is called **fair** if the following two conditions are fulfilled:
(i) there exists a strategy S_O of the player, such that - regardless of what the holder puts into the envelopes - it is true that the player's expected win $E(S_O)$ from buying satisfies $E(S_O) \geq Q$
(ii) There exists a strategy for the holder, such that the player's expected win $E(S)$ from buying does not exceed Q regardless of which strategy S the player chooses.

Observe that condition (ii) does not imply that $E(S)$ always has to exist. We have proved:

Theorem 5. *If the holder and the player agree on a lower bound L for the values in the envelopes, then $L/4$ is a fair price.*

Note that as soon as the vendor puts a value x in the envelope which is not of the form $2^k L$, the player has an advantage.

One might argue, that the holder's optimal strategy (it is the only one) is unsuitable for practical use because the distribution of X_H is not finite and even has infinite expected value. The holder thus has to cut off the 'tail' of this distribution and must 'hide' the remaining probabilility-values somewhere. This makes him vulnerable to a lucky guess about X_H by the player. While this is true, we have already shown that the player's strategy S_1 exactly has expected value $L/4$ as long as x is of the form $2^k L$. Thus, in order to achieve a higher gain, the player himself must turn to another strategy. But it is easy to see that the described strategy is the only one that guarantees him an expected value of at least $L/4$. Thus, the player himself becomes vulnerable if he tries to exploit the fact that the holder is assumed to have only a finite amount of money.

A second approach to achieving fair conditions for both the player and the vendor can be made in the form of an additional charge to price agreement G_1 which is proportional. Let α be the proportionality factor of this charge. The player would only pay this charge if he buys and would then pay an amount of $\$(1 + \alpha)x_1$.

Fairness can be defined analogously to the above definition. Of course, to obtain fair conditions, α must be large enough to compensate for the fact that the holder gets nothing unless the player buys. Suppose the player acts according to S_1 with $p = 2/3$ and suppose he finds $\$x_1$ with $x_1 = 2^k L$. Then he has an expected gain of $L/4$, but will buy only with probability $P_A(x_1) = 2^{-k}$, which forces $2^{-k}\alpha x_1 \geq L/4$ and hence

$$\alpha \geq 1/4$$

Surprisingly, this value does not depend on L. One might therefore suspect that $1.25 \cdot x_1$ is a fair price for the second envelope. But this is not true. In contrast to the case where a fee must be paid in all situations, the player here is not forced to have a minimum expected gain for each choice of the vendor. He can therefore choose a very restrictive strategy of buying and may thus select an S_1-strategy with parameter p as close to 1 as he wishes. In the extreme, this would mean that he only buys when he finds an L-primitive amount of money in the first envelope. These arguments show that the only possible fair way of choosing α is $\alpha = 1$ and this, in turn, makes the game completely uninteresting for the player.

4.2. No lower bound assumed. We will evaluate the same possibilities as in 4.1: a constant fee and an additional proportional charge.

The result of the first possibility is immediately clear: A fair constant price cannot exist because the vendor might put an amount of money in the envelopes that is less then the amount of the fee. Thus, we are left with the second possibility: an additional charge with proportionality factor α.

The key to this approach are the following observations: Let α be given and define

$$\tau = \frac{1 + 2\alpha}{1 - \alpha}$$

Then, if the holder puts $\$x$ and $\$2x$, respectively, in the envelopes, the player is not at a disadvantage if and only if

$$P_A(x) \geq \tau P_A(2x) \tag{3}$$

On the other hand, if the holder chooses the distribution of X_H such that for a real number x the inequality

$$P_H(x/2) \geq (2/\tau) P_H(x) \tag{4}$$

holds, he need not be concerned if the player finds $\$x$ first (at least in state Z_1). So, α is fair if and only if (3) and (4) hold for all possible values of x. (Note: If X_H is a continuous random variable, condition (4) must be replaced by $d_H(x/2) \geq (4/\tau) d_H(x)$ where d_H is the density function of X_H).

Now, (4) can be fulfilled for all x iff $\alpha > 1/4$. But for any α with $0 < \alpha < 1$ the player's condition (3) can only hold for all x if $P_A(x) = 0$ for all x, which means that the player must never buy. Thus, fairness in the sense of the beginning of this chapter can only be obtained in a way that kills the game. We shall, therefore, look for an understanding of fairness that allows for games which are less dull:

It is most likely that both opponents have their own private (and unknown to the other) **level of substantiality** for an amount of money, i.e. if the value in the envelopes is below this level they do not really mind if they loose. But it is important to them to have fair conditions whenever the money in consideration exceeds this level. So we define:

Definition 6. *A price agreement $Pr(x_1) = (1+\alpha)x_1$ is* ***substantially fair*** *if*
i) for each level of substantiality $L_P > 0$ of the player there exists a player's strategy S that guarantees that the conditional expected value

$$E_P = E(X_B - Pr(X_A)|\, X_H \geq L_P \text{ and } S \text{ tells to buy})$$

satisfies $E_P \geq 0$ for all strategies X_H of the vendor and
ii) if for each level of substantiality $L_H > 0$ of the holder there is a strategy X_H with the analogous property that

$$E(X_B - Pr(X_A)|\, X_A \geq L_H \text{ and } S \text{ tells to buy}) \leq 0$$

for all strategies S of the player.

Since substantial fairness is a weaker condition than fairness, a fair α is automatically also substantially fair. In fact, any positive α is substantially fair as both competitors can achieve substantially fair games in a cheap way: The holder might offer only amounts of money which are below his level of substantiality and the player might always pass (choose not to buy the second envelope). Both of these ways of playing the game are to a large degree unwelcome and should thus be excluded from consideration. To do this we define the following:

By the **order of performance** of a player's strategy S we understand the maximal $\epsilon > 0$ such that $\liminf_{x\to\infty} P_A(x) \cdot x^{1/\epsilon} > 0$. For instance, the strategy S_1 with parameter p has order of performance $1/log_2(p/(1-p))$. The order of performance of a strategy is a measure how frequently the strategy will tell the player to buy the second envelope. Let us call a player's strategy S **performance-essential** if its order of performance is at least 1. Let us further call a substantially fair price agreement **game-encouraging** if all the player's strategies, as required by definition 6, can be chosen to be performance-essential. From (3) we conclude: α is game-encouraging if and only if $\alpha \leq \frac{1}{4}$.

Now let us analyze the vendor's situation: He may accept that he will sometimes have to assume a higher risk by offering unsubstantial amounts of money in order to ensure that his chances are fair when the amounts are substantial, but he will not be pleased if he has to do so very often. So we define:

A substantially fair price agreement is called **unrestrictive** if, whatever his level of substantiality L_H may be, the vendor can fulfill the requirements of definition 6 by a strategy such that the probability of an unsubstantial value of X_H becomes arbitrarily small. By (4) it follows that $\alpha < \frac{1}{4}$ is not unrestrictive, since in this case we have $2/\tau > 1$. But larger values of α are unrestrictive: Let $\epsilon > 0$ be given and choose a natural number k such that $1/k < \epsilon$. Now define X_H to be equally distributed on the set $\{L_H/2, L_H, 2L_H, ..., 2^{k-1}L_H\}$.

To sum up, we have proved

Theorem 7. *The price agreement $Pr(x_1) = 1.25x_1$ is substantially fair, unrestrictive and game-encouraging. It is the only price agreement of the form $Pr(x_1) = (1 + \alpha)\, x_1$ with these properties.*

Finally, we have come to the conclusion that $\alpha = 1/4$ is the correct proportionality factor for an additional charge. Only this value of α can make the game attractive to both the holder and the player. Although the arguments of the 'naive subjectivist', cited at the beginning of this article, are miles away from being correct, $\alpha = 1/4$ is exactly the value he would have chosen at once! But there is no reason for him to rejoice - it must be kept in mind, that the player can only be sure to have approximately even odds if he applies an appropriate mixed strategy.

We end this paper with some remarks:

Remark 2. *i) The substantiality-levels in the definition of substantial fairness may seem to be meant as mathematical 'small ϵ'. They indeed can be interpreted in this sense, but there is no need to consider them this way. In practical application, they may be thought of as the real values at which the respective competitors start to take the game seriously. Those involved should not put these levels too low. For instance, suppose envelopes of the described type would be offered at some sort of stock-market and someone would try to act as an arbitragist by looking out for envelopes offered at a price less than $x_1 + 25\%$. A small level of substantiality implies a small performance-rate which implies less gain and a comparatively high deviation.*
ii) The definition of 'game-encouraging' may appear somewhat arbitrary: why take 1 as the borderline? There is no striking reason except that 1 is a nice, round number. It is in any event true that $\alpha = \frac{1}{4}$ maximizes the order of performance among all unrestrictive price agreements.
iii) The two envelope paradox, as represented here, may be viewed as an example of a zero-sum game with an infinite number of pure strategies.

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5. Nachwort zur Entstehungsgeschichte

Die Entstehungsgeschichte dieses Artikels ist mit dem Standort Friedberg so eng verknüpft wie es wahrscheinlich nur bei wenigen Artikeln der Fall ist. Auf das Zwei-Umschläge-Paradoxon wurde ich von Manfred Börgens, der den Artikel von Linzer gelesen hatte, am Friedberger Fachhochschultag 1994 hingewiesen. Im Anschluss daran entwickelte sich unter den in der Widmung genannten Kollegen eine lebhafte, sich bis zum Semesterende hinziehende (Pausen-)Diskussion über das Thema. Als Fazit der Diskussion aus meiner Sicht entstand dieser Artikel, der in einer ersten mit der vorliegenden Arbeit fast identischen Version bereits im Februar 1995 vorlag. Ich habe den Artikel damals bei AMM eingereicht, wo er postwendend und ohne Prüfverfahren mit der Begründung abgelehnt wurde, dass man (sinngemäß) aus Balancegründen diesen Themenbereich momentan nicht berücksichtigen könne. Diese doch recht entmutigende prompte Antwort habe ich damals nicht so recht verstanden, denn mit Linzers Artikel ist die Thematik beim besten Willen noch nicht ausgeschöpft.

Heute kann ich das Verhalten von AMM etwas besser verstehen, denn ich weiss, dass ich nicht der einzige war, der seinerzeit einen Artikel über das Thema dorthin schickte und vermute, dass als Reaktion auf Linzers Artikel eine ganze Reihe von Beiträgen eingereicht wurden, denn es gab zu der Zeit schon einige Arbeiten zu dem Thema in den verschiedensten Zeitschriften. Diese sind in Linzers Arbeit nicht angegeben und waren mir und den Friedberger Kollegen als Nichtspezialisten dieses Gebiets (im engeren Sinne) nicht bekannt. Erst sehr viel später wurde ich darauf aufmerksam gemacht. Unten findet man (ohne Anspruch auf Vollständigkeit) eine Auflistung weiterer Artikel zu dem Thema, von denen einige älter sind als 1994. Die meisten von ihnen behandeln das ursprüngliche 'Paradoxon' oder verwandte Problemstellungen wie z.B. das 'Wallet-Paradoxon' (zwei Personen vergleichen den Inhalt ihrer Brieftaschen; derjenige, der weniger Geld darin hat, erhält die andere), einige gehen aber auch darüber hinaus. Die von mir gefundene C_{max}-Strategie ist nicht neu, sie wird u.a. in [1] und [2] ebenfalls beschrieben: die Frage nach einer möglichen Strategie und die Beschreibung der Strategie gehen nach Auskunft von F. Thomas Bruss vermutlich auf Thomas Cover (1987, keine zitierbare Quelle bekannt) und Stephen Samuels [7] (1991) zurück. Die Fragestellung des richtigen Preises des zweiten

Umschlags wird von Bruss in [2] ebenfalls angeschnitten (in einer allgemeineren Fragestellung), aber nicht beantwortet. Meine Überlegungen zum fairen Preis (oder Gleichgewichtspreis) des zweiten Umschlags (Abschnitt 4.2) mit der Pointe, dass der eigentlich falsche Preis doch der richtige ist, finden sich aber in keiner mir bekannten Quelle.

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