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Simple and Good - Option Pricing with Regime Switching Skew Tree Models

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1 Introduction

This paper deals with models of share price movements for the purpose of evaluating options. Two practical aspects are particularly important here: the feasibility of the models and the testing of real option prices in "normal" and also in not entirely quiet times.

Since my practical and research semester 2007/2008, in cooperation with Kathrin Diener and others at the BHF-Bank, I have been working on models for stock price developments during my available time for research purposes. Early designs can be described as model-free approaches [Hausmann 2007]. These were investigated in two diploma theses in 2008 in comparison to the local volatility model [Dupire 1994] and the model of Heston [Heston 1993]. The investigations showed encouraging aspects, but overall these models in their implemented form were inferior to the two mentioned models (for different reasons). Overall, however, the model design from 2007 has not yet been fully fathomed and cannot be regarded as conclusive.

As a result of 2007/2008, however, there was a shift of my interest towards more structured price models. Furthermore, the spectrum of the models considered was extended to include jump diffusion processes and pure jump processes. The variance-gamma process was investigated as a pure jump process. A master thesis from the year 2016 [Kraliczek 2016] had the interesting result that the variance-gamma model fits well to the real DAX® price development, but not to the DAX® option prices of Eurex (this is no contradiction!). This was an important basis for my decision to turn to regime switching models for my last research term (winter semester 2016/17). With such models, there is a random change between several usually simple processes (regimes), which basically run parallel. This change is controlled by a (hidden) Markov chain. As in 2007, discrete models were chosen.

In the winter semester 2016/17, the general system framework was defined and its possibilities were explored. This concerns on one hand the theoretical possibilities (in particular with regard to the representable implied volatility surfaces (IVS)), but on the other hand also the practicability using a computer. An important result is certainly that it was possible to realize models with quite extreme parameter values on standard laptops in such a way that sufficiently accurate results (option prices) were obtained.

Towards the end of the research semester, several tests were already carried out using real option prices, the results of which were encouraging. The presentation of the results of the research semester can be found in [Hausmann 2017] in German, but they also form (in English) the first part of this paper (with the exception of studies on some singular trading days in [Hausmann 2017])[1].

In the first part of this article a model construction kit for option price models is presented, whereby the stable distribution of the Markov chain, which controls the change between the regimes, also plays a certain role.

From the outset, however, it was planned not to be content with just a few random tests of real data, but to systematically measure the performance of the models over extended time periods. This happened almost exclusively after the winter semester 2016/17. The results achieved can be found in part II of this paper, which additionally contains a more detailed description of the special model type used and a few conceptual additional developments. The presentation of the results of this empirical investigation is the main part of this paper.

The phrase "simple and good" of the title needs explanation, since it sounds more like an (extremely simple) advertising slogan than a suitable title for an article that sees itself as the elaboration of a scientific investigation.

Let's start with "simple": The models presented are simple in structure and easy to program, although there is the problem of tree size (calculability). Above all, however, the models are simple in terms of mathematical conceptualization, since they are models of finite type, i.e. all occurring

[1] The elaboration announced under the name 'Skew Trees and Variable Time Steps: A Model Kit for Option Pricing' is therefore superfluous and no longer planned.

probabilities are probabilities of finite spaces. For these models, the fundamental theorems necessary for option valuation apply to the same extent as for continuous and time-continuous models, but the important terms such as *equivalent probability measure* are fundamentally much easier to understand. The situation is the same as with the Black-Scholes-Merton model on one hand and the Cox-Ross-Rubinstein model on the other hand.

Of course, with discrete models for option pricing one rarely thinks of only one model, but typically of a whole class of discrete models, that all deliver almost the same prices of relevant options and ideally all approximate a single continuous model. But it is not necessary to understand a discrete model only as a numerical approximation of a continuous model. Once one has decided on a discrete model in a given situation, one moves into a cosmos in which concepts such as *arbitrage* (*freedom*) and *self-financing trading strategies* with essentially the same expressiveness exist independently. And this cosmos is much simpler than that of a continuous model.

The "good" reflects the experience of the investigations in part II concerning Eurex DAX® call options during the Brexit vote in 2016 and the French presidential elections in 2017. To approximate the market prices (on average more than 400 per trading day), a model type was used that can be described as extremely simple (thus again "simple") and requires only two components (= regimes) of the model kit. From the beginning it was surprising how well the IVS found on the market can be represented by this model type. Although the performance of the local volatility model in this respect was not quite achieved, the regularity of the processes is much higher, which gives hope for better dynamics. In "normal" times within the investigation periods even suitable so-called steady-state models could be found (see Chapter 7.2, esp. the figures on page 61). In these models, the logarithm of the discounted share price has stationary (but not independent) increments. With the "unusual" trading days as above (all the days shortly before and after the Brexit vote as well as the days close to the first ballot in France) well fitting models could only be found if the model did not also assume a stable market condition.

The characterization as good, but not as excellent or even perfect results from the details of the 2nd part. An "excellent" model from the user's point of view would be one that allows approximately the same system parameters for all trading days and that simply swallows the turbulent days. Such a model would allow completely unproblematic hedging. The model type investigated (and any model I know) is far from this, but the results of the rudimentary investigations on stability (sections 7.4 and 8.3) could be an indication that the usual practice of "always evaluating everything with the current model and readjusting the hedge portfolio accordingly" should also work well with the model type ("good"). However, this needs to be investigated much further. Above all, a hedging strategy appropriate to the model still needs to be developed.

I would like to thank the students of the mathematics master program at the THM Holger Schubert and Sabri Dali for their support in obtaining market data. I would also like to thank the MND department of the THM and especially my mathematics colleagues for making it possible for me to spend one more research semester in my last term before retirement. Finally, last but not least, I would like to thank Nils Koop very much for his help with the formulations in English.

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Part I
A Skew Tree Model Kit for Option Pricing

2 Introduction to the First Part

In this first part special tree models for option pricing are introduced, including aspects of the realization. The models are *regime switching models*. More precisely: 3 types of regimes are presented, which can be assembled in a modular way. In this way, classical option price models such as jump diffusion models (cf [Cont Tankov 2004]) or stochastic volatility models ([Heston 1993] and others) can be approximated (the even more classical Cox-Ross-Rubinstein model is a special case), but it is also possible to generate models which, to my knowledge, have not yet been thoroughly investigated. This is especially true for regimes called *skew ramifications* if very disparate parameters are used.

Overall, regime switching models for option pricing have been known for quite some time, including tree models ([Aingworth et al 2006] or [Liu and Nguyen 2015], also see [Derman 1999]), so this is not a completely unknown area. In this article, however, particular emphasis is placed on the real applicability. This means that tree models are aimed at that can (at least) value options with a maturity of one day to about one year well, i.e. approximate the real option prices convincingly. This should be done with a single tree. This is necessary if exotic options are to be valued with the help of the tree.

Designing such trees in theory is not a problem, but the realization can be an issue. The resulting trees can become too large to perform extensive calculations in them. This is especially true when the implied volatility surfaces observed on the market often require extreme parameter values.

Special tools are needed to overcome these problems. It starts with the use of *lattices*, which are well known tools to obtain recombinant trees in an extended sense. But that alone is not enough. *Varying time steps* and *tree-cutting* are additional tools, which all together create the possibility to realize models for which this seemed impossible at first. All the models realised proved to run on a laptop more than five years old.

Part I is structured as follows: First, the construction kit is described in general terms (chapter 3), which also includes an explanation of some aspects of the Markov chains used. Chapter 4 contains a description of the implementation and the tools used to obtain computable trees. Finally, Section 4.2 presents some typical models and representable implied volatility surfaces.

3 The Model Kit

3.1 What is Modelled - and How is it Done?

The dynamics of the discounted price $\widetilde{S}_t$ of an asset S - a stock or a stock index for example - are modelled as a stochastic process under the price determining equivalent martingale measure $\mathbf{Q}$. All models occuring are of finite type (or tree models), i.e. every bounded time interval contains only a finite number of trading times t_i and at each t_i there is only a finite number of possible values of $\widetilde{S}_{t_i}$. 'Discounted' means discounted with respect to the riskless interest rate, which is therefore assumed to exist. The riskless interest rate may be time dependent, but not stochastic. Without loss of generality all models start at $t = 0$.

The demand on $\mathbf{Q}$ to be a price determining martingale measure means that the process of the discounted price $\widetilde{CC}$ of every (traded) contingent claim CC of S is a martingale under $\mathbf{Q}$. As S itself is a contingent claim of S, it follows that the process $\left(\widetilde{S}_t\right)$ is a martingale.

A further consequence is that the value of an option of European type can be expressed as expected value. Let $\widetilde{CC}_e$ be such an option with maturity $T > 0$. Then for each $t_0 < T$ the discounted value $\widetilde{CC}_e\,(t_0)$ of CC_e at t_0 is the conditional expected value of the payoff at maturity date:

$$\widetilde{CC}_e\,(t_0) = \mathcal{E}_Q\left(\widetilde{CC}_e\,(T)\,|\,\mathcal{F}(t_0)\right) \tag{1}$$

Here $\mathcal{E}_Q$ denotes the conditional expected value with respect to $\mathbf{Q}$ and $\mathcal{F}(t_0)$ is a σ-algebra that represents the information known at t_0. In finite discrete models the information known at a point in time t_0 can less formally be described as the knowledge of the path the system had taken up to t_0. As to the models considered here, 'path' need not be the same as "development of $\widetilde{S}_t$" (cf the end of this section).

All discounted values refer to discounting to $t = 0$, but formula (1) remains true if $0 \leq t \leq t_0$. The formula in general requires some boundedness conditions for the payoff of the contingent claim (as a random variable) which are always fulfilled for models of finite type.

For $t_0 = 0$ the conditional expected value of formula (1) becomes an ordinary expected value:

$$CC_e\,(0) = \widetilde{CC}_e\,(0) = \mathbf{E}_Q\left(\widetilde{CC}_e\,(T)\right) \tag{2}$$

The probability measure $\mathbf{Q}$ is almost never appropriate to describe the process of $\widetilde{S}_t$ in reality, but it should be equivalent to a probability measure $\mathbf{P}$ which suits the real-world process. The general theory says that, starting from the real world measure $\mathbf{P}$, the existence of a measure $\mathbf{Q}$ is equivalent to the absence of arbitrage possibilities in the system (*first fundamental theorem of asset pricing)*, cf [Cont Tankov 2004] for a general formulation or [Hausmann Diener Kaesler 2002] for an elementary proof for finite models. If $\mathbf{P}$ defines a *complete* model (or market) (cf any book on mathematical finance), $\mathbf{Q}$ is uniquely determined (*second fundamental theorem of asset pricing)*. The regime switching models of this article are complete if and only if they consist of only one regime, i.e. if there is no switching at all. So all interesting models are incomplete. This means that the prices of contingent claims of S are not uniquley determined by the real world process of S and the requirement that there are no arbitrage possibilities. A certain vacuity remains in the price building process that is filled by the traders (thereby following whatever principles). Anyway, the first fundamental theorem guarantees that in the absence of arbitrage possibilities a price determining equivalent martingale measure does exist.

Remark 1 *'Complete' is often (just as above) used in the meaning, that the asset S in combination with risk-free lending and borrowing is a complete system. But complete systems can have more than one risky asset. At least in models of finite type, it is always possible, to complete an arbitrage-free system by "declaring" a suitable finite set of derivatives to "genuine" assets. Then the processes of S (or $\widetilde{S}$) and the prices of these derivatives uniquely determine* $\mathbf{Q}$ *(cf [Hausmann Diener Kaesler 2002] for example; illuminative terms in this context are 'implied trees' and 'Arrow-Debreux securities'; also see section 5.4). In practice, contingent claims, for which there is a liquid market, are reasonable candidates for genuine assets.*

The models that are covered in this article are *regime switching models.* In every point in time one of a finite set of states, so called *regimes*, is active and sets the instantaneous rules for the development of $\widetilde{S}_t$. The change from one regime to another happens all at once and is guided by an ergodic Markov chain. This chain is *hidden*, i.e. it cannot be seen directly by a naive observer (whereas S_t, the current value of S, can). The situation of the system at any one time t is completely described only by the pair $\left(\widetilde{S}_t, Z_j\right)$, where Z_j is one of the regimes, not by $\widetilde{S}_t$ alone. Within the models, the instantaneous regime is at any time part of the known information.

This is similar to the situation in stochastic volatility models like the model of Heston [Heston 1993], where the instantaneous volatility can be considered as an active regime, that also cannot be seen directly. When looking at Heston's model this way, one sees a model that is even able to handle infinitely many (volatility-)regimes. But a regime can be more than just a certain volatility. Moreover, in Heston's model volatility changes continuously.

3.2 Regimes

Regimes and Regime Switching Models were originally introduced to financial modelling by Hamilton ([Hamilton 1988] and [Hamilton 1989]) and find most of their applications in time series analysis (cf [Ang Timmermann 2012]). In this article the term *regime* has a very restricted meaning as certain binary ramification in a tree model[2]. They describe the possible development of the entity, the dynamics of which are modelled (here $\widetilde{S}_t$) in a time interval $[t_i, t_{i+1}]$. As time steps within one model may have different lengths, the

[2]There is one exception. In chapter 5.3 regimes are understood in a slightly more general meaning.

definition of a regime must specify the possible developments of the modelled entity over intervalls $[t_i,\, t_{i+1}]$ of various lengths $\Delta_t = t_{i+1} - t_i$.

3.2.1 Regimes of CRR Type (BSM Type)

Regimes of this type stipulate branchings like

$$\begin{array}{ccc} & & u\,\widetilde{S} \\ & \nearrow_{q} & \\ \widetilde{S} & & \\ & \searrow^{1-q} & \\ & & d\,\widetilde{S} \end{array}$$

with $u = e^{\sigma\sqrt{\Delta_t}}$, $\sigma > 0$ and $d = 1/u$. These ramifications are well known from the *Cox-Ross-Rubinstein model* (*CRR model*), a model that resembles the Black-Scholes-Merton model (BSM model) in the class of binary tree models (cf [Black Scholes 1973], [Cox Ross Rubinstein 1979]).

The probabilities q and $1-q$ can be determined from the requirement that the process $\left(\widetilde{S}_t\right)$ has to be a martingale, which implies

$$q\,u + (1-q)\;d = 1$$

and hence

$$q = \frac{1-d}{u-d} \tag{3}$$

u, d and q thus depend on the volatility σ and the length Δ_t of the time interval $[t_i,\, t_{i+1}]$ (which may be variable). The regime thus is characterized by σ.

3.2.2 Skew Ramifications

These are regimes as in the preceding section, the only difference is $u \neq d$. In principle, these regimes are still of CRR type and generate recombining binomial trees, although the CRR models used in practice mostly have $u = d$. It takes two positive numbers σ_u and σ_d to characterize such a regime. u and d are given by

$$u = e^{\sigma_u\sqrt{\Delta_t}} \qquad d = e^{-\sigma_d\sqrt{\Delta_t}}$$

In these regimes, too, the martingale property of the stochastic process uniquely determines the ramification probabilities. Formula 3 is still valid.

The skew ramifications that appear in this article mostly have $\sigma_d >> \sigma_u$, for example $\sigma_u = 0.01$ and $\sigma_d = 5$. Then at least for small time steps

Δ_t the probability q of an upward move is close to 1. This is because of $q \approx \sigma_d / (\sigma_u + \sigma_d)$.

BSM (CRR) regimes and skew ramifications will be called to be of *diffusion type.*

3.2.3 Jump Ramifications

Jump ramifications are fundamentally different from regimes of diffusion type: The height of the jump in $[t_i,\, t_{i+1}]$ is independent of the length of the time interval. A constant *jump factor* j_p tells how the value of $\widetilde{S}$ changes, if a jump occurs (for instance $j_p = 0.7$ stands for a loss of value of 30%). In principle, jumps can be upward or downward jumps, and both kinds of jumps can be modelled in almost the same way, but for this study only downward jumps have been considered. The corresponding event in real life is a slump or crash event in $[t_i,\, t_{i+1}]$ that dominates all further moves in this time interval, and that happens only with low probability. It is assumed that this probability is so low, that the possibility of more than one crash event in one time interval can be neglected.

Not the height, but the probability of a jump event in $[t_i,\, t_{i+1}]$ shall depend on the length Δ_t of the time interval - just as it is with a Poisson process. Poisson processes are continuous-time stochastic processes, but can be approximated by time-discrete processes. For a Poisson process the frequency of jump events is characterized by a positive real number λ, which is called the *jump intensity.* λ has the property, that the probability of no jump event in a time interval of length Δ_t is

$$q_u = e^{-\lambda\,\Delta_t} \tag{4}$$

and this is how it shall be with jump ramifications. What should happen to $\widetilde{S}$ in the situation of no crash occuring? The intention is to use a binary ramification, so a positive number u can be defined by the equation $\widetilde{S}_{t_{i+1}} = u\,\widetilde{S}_{t_i}$.The indispensable requirement that the process $\left(\widetilde{S}_t\right)$ is a martingale then leads to

$$q_u\,u + (1 - q_u)\,jp = 1$$

and hence

$$u = \frac{1 - (1 - q_u)\,jp}{q_u}$$

u depends on Δ_t, but not on $\widetilde{S}_t$. The ramification looks like this:

$$\begin{array}{ccc} & & u\,\widetilde{S}_t \\ & \nearrow^{q_u} & \\ \widetilde{S}_t & & \\ & \searrow^{(1-q_u)} & \\ & & jp\,\widetilde{S}_t \end{array}$$

3.3 Time Steps and Regime Switching

As already indicated, the models introduced in this article, in which calculations are executed, are finite models. There is a starting point $t_0 = 0$, an end point T and there are finitely many more points in time $t_0 < t_1 < ... < t_{n-1} < t_n = T$.

It is not assumed that $\Delta_{t_i} = t_{i+1} - t_i$ is the same for every i. Far from it: All models start with small time steps, and, as i increases, Δ_{t_i} eventually gets bigger. This need not happen in a monotone way. Smaller steps can be included again at any time. This is often necessary to make a certain date (a maturity mostly) a moment t_i of the model.

The use of varying lengths of time steps has no fundamental meaning. There is just the pragmatic purpose of obtaining models that produce suitable prices for short running as well as for long running options and that can be dealt with on standard laptops. Every model with varying time steps is always supposed to be a good approximation of the corresponding model with the smallest occuring time step as constant time step. This will be covered in more detail in the next chapter. Prior to covering this, it is important to note the following additional features of the models:

To each point in time t_i there are finitely many *states* $s(i,j)$, which are characterized by

1. the price (value) of the quantity $\widetilde{S}$ that is modelled by the system; $\widetilde{S}$ is thought of as the discounted price (with respect to t_0) of an asset paying no dividends

2. the *active (ruling)* regime Z, that sets the rules for the development of $\widetilde{S}$ in the next time step from t_i to t_{i+1}.

The total number of regimes (for all i) is assumed to be finite.

To describe the development of the system (not only $\widetilde{S}$) in one time step, the active regime needs to be defined. Is Z going to stay active or will it be replaced by another regime?

Let us assume that all time steps have the same length Δ_0. In that case the transition between the regimes shall follow the rules of a finite homogeneous ergodic Markov chain - independently from $\widetilde{S}$:

Let $Z(t_i)$ denote the active regime at t_i which is one of l possible regimes $Z_1,\ldots, Z_l$. Then the conditional probabilities

$$a_{k\,j} := \mathbf{Q}\left(Z(t_{i+1}) = Z_j \,|\, Z(t_i) = Z_k\right)$$

are the same for each i (homogeneity) and are not influenced by the value of $\widetilde{S}_{t_i}$ or the development $\widetilde{S}_{t_i} \to \widetilde{S}_{t_{i+1}}$ or the complete path of $\widetilde{S}$ or Z up to t_i. The $a_{k\,j}$ are called *transition probabilities* of the Markov chain, the matrix $A = (a_{k\,j})$ is the *transition matrix.* The requirement of ergodicity guarantees that each regime can be reached from each regime in a finite number of steps (with postive probability) and that there are no static periods for this (for a formal definition of ergodicity see any book that covers finite Markov chains, for instance [Winston 2003]).

To obtain the possibility of using variable lengths of time steps and still remain in the frame of finite Markov chains, the following condition is imposed on the lengths Δ_{t_i} of time steps of a model:

Claim. *There is a smallest time interval length* $\Delta_0 > 0$*, such that every* Δ_{t_i} *(*$i = 0,\ldots, n-1$*) is an integral multiple of* Δ_0.

Under this assumption it is always possible to refine the time pattern of the model in such a way that all time steps have length Δ_0. If A is the transition matrix of an interval of length Δ_0, the *Chapman-Kolmogorov equations* tell that the transition matrix of a time interval of length $m\,\Delta_0$ has to be A^m. So A determines the transition matrix of every time step $t_i \to t_{i+1}$ of the original model, if the original and the refined models are supposed to fit together. The regime switching of the original model with a varying length of time steps, i.e. the process $(Z\,(t_i))$, in this case is an inhomogeneous Markov chain with homogeneous refinement.

The two-dimensional process of the pairs $\left(\widetilde{S}_{t_i},\, Z(t_i)\right)$ is also a discrete Markov chain (but with infinitely many states, if the set of the t_i is not finite). The possible values of $\left(\widetilde{S}_{t_{i+1}},\, Z(t_{i+1})\right)$ and the probabilities

$$\mathbf{Q}\left(\left(\widetilde{S}_{t_{i+1}},\, Z(t_{i+1})\right) = (Y, Z_j)\,\middle|\,\text{path from } \left(\widetilde{S}_{t_0},\, Z(t_0)\right) \text{ to } \left(\widetilde{S}_{t_i},\, Z(t_i)\right)\right)$$

only depend on $\left(\widetilde{S}_{t_i},\, Z(t_i)\right)$. The resulting process $\left(\widetilde{S}_{t_i}\right)$ however does not have the Markov property, if two or more different regimes occur.

Augmentation. The following slight extension of the models increases their ability in a reasonable way. Instead of assuming that the system starts with a pair $\left(\widetilde{S}_{t_0}, Z(t_0)\right)$ in t_0, one can artificially divide the start of the process in two parts: At the beginning there is only the price $\widetilde{S}_{t_0} = S_{t_0}$ of the asset, the determination of the active regime follows in an almost timeless (0th) step. This can be justified by the fact that the ruling regime is not as clearly visible as the market prices, but there are hints. At the beginning there is only a probability distribution $(q_1, \dots, q_l)$ on the set of possible regimes, the *initial distribution* or *augmentation.* In the 0th step the active regime is selected - guided by this distribution.

Any model, which makes use of this extension, will be called a *general augmented model.* Of course, by the trivial distributions $(0, ..., 0, 1, 0, ..., 0)$ the models with a fixed active regime at the start are included in this definition. These are also called *pure models.*

There is another distribution of special interest. Finite homogeneous ergodic Markov chains $\mathfrak{X} = (X_n)_{n=0,1,...}$ converge to a uniquely determined distribution $(\pi_1, ..., \pi_l)$ of its states $Z_1, ..., Z_l$, i.e. for all $1 \leq i \leq l$ the equation

$$\lim_{n \to \infty} \mathbf{P}\left(X_n = Z_i\right) = \pi_i$$

holds. This is true for every initial distribution, i.e. distribution of X_0. $\pi = (\pi_1, ..., \pi_l)$ is called the *steady-state* (or *equilibrium* or *stationary*) distribution of $\mathfrak{X}$. The uniquely determined steady-state distribution has the property, that once reached, it will never be changed again. The transpose of the row vector $(\pi_1, ..., \pi_l)$ is an eigenvector with eigenvalue 1 of the transpose of the transition matrix A of the process. Equivalently

$$(\pi_1, ..., \pi_l)\ A = (\pi_1, ..., \pi_l)$$

Moreover, the powers of A converge to a package of vectors π:

$$\lim_{n \to \infty} A^n = \begin{pmatrix} \pi_1 & \dots & \pi_l \\ \dots & & \dots \\ \pi_1 & \dots & \pi_l \end{pmatrix} \tag{5}$$

As to our skew tree models, a special situation is given, if the initial distribution of the regimes equals the steady-state distribution:

$$(q_1, ..., q_l) = (\pi_1, ..., \pi_l)$$

General augmented models, in which this equation holds, will be called *steady-state models.* In these models the equilibrium distribution of the regimes is given for every point in time t_i. In general, this distribution is no longer given, if an additional condition like a special value of $\widetilde{S}_{t_i}$ is imposed. But for all regimes Z introduced in section 3.2 the conditional distribution of $\log\left(\widetilde{S}_{t_{i+1}}/\widetilde{S}_{t_i}\right)$ under the condition, that Z is active, does not depend on $\widetilde{S}_{t_i}$. From this it follows:

Theorem 2 *In a steady-state model built from a finite number of regimes as introduced in 3.2.1 to 3.2.3 the distribution of* $\log\left(\widetilde{S}_{t_{i+1}}/\widetilde{S}_{t_i}\right)$ *only depends on* $\Delta_i = t_{i+1} - t_i$. *Hence, if all* Δ_i *are equal, the distribution of* $\log\left(\widetilde{S}_{t_{i+1}}/\widetilde{S}_{t_i}\right)$ *is the same for all* i, *which means that the stochastic process* $\left(\log\left(\widetilde{S}_{t_i}\right)\right)$ *has stationary increments.*

But it must be kept in mind that the increments are not mutually independent. This is good news, not bad news, because otherwise the central limit theorem could be applied and would imply convergence to a lognormal distribution of $\widetilde{S}_{t_{i+1}}$ as $i \to \infty$. Even without independence the property of stationary increments adds regularity to a process, which normally is advantageous. General augmented models are not completely irregular. At least they always have asymptotically stationary increments, which follows from formula (5).

Remark 3 *In the context of the skew tree models of this article, the equilibrium distribution* π *under some aspects plays the role of the mean reversion of volatility in other models of stochastic volatility.*

General augmented models, that only differ by their initial distribution, are very closely related and may be considered as members of the same class in some sense. If each initial probability q_i is positive for both, their trees - nodes, possible transitions and transition probabilities - only differ by their 0-th transition probabilities q_i. Thereafter the trees are identical.

Theorem 4 *a) General augmented models, which only differ by their initial distribution and are equal elsewhere, have the same internal prices (prices at nodes* $\left(\widetilde{S}_{t_i},\, Z(t_i)\right)$*) of European options after the 0-th step. This is true for all nodes* $\left(\widetilde{S}_{t_i},\, Z(t_i)\right)$*, that occur in both models.*
b) In a skew tree model with regimes $Z_1,\ldots,\, Z_l$ *let* $C_1,\, \ldots,\, C_l$ *be the prices of a European option in the respective pure models. Then the price* C *of this option in a general augmented model with initial distribution* $(q_1, \ldots, q_l)$ *is*

$$C = \sum_{i=1}^{l} q_i C_i \tag{6}$$

Proof. a) If, as described before the theorem, the q_i of all considered models are > 0, this is an immediate consequence of formula (1). Otherwise the conditional expected value of this formula applied to a node $\left(\widetilde{S}_{t_i},\, Z(t_i)\right)$ only depends upon the subtree generated by the paths starting at this node.
b) This follows from the formulae (1) and (2). Let $\widetilde{CC}_e\,(T)$ be the discounted value of the option at maturity T. Then

$$\mathbf{E}_Q\left(\widetilde{CC}_e\,(T)\right) = \sum_{i=1}^{l} q_i\,\mathcal{E}_Q\left(\widetilde{CC}_e\,(T)\,|\,Z(t_0) = Z_i\right)$$

■

If there are m European options and $C_j^{(m)}$ is the vector of their prices in the j-th pure model ($j = 1, ..., l$), then part b of the theorem implies that the set of price combinations that can be realized by a general augmented model of the same class, equals the convex hull of the $C_j^{(m)}$ in $\mathbb{R}^m$.

Remark 5 *The skew trees of two finite general augmented models, that only differ by their augmentation and have the same set of nonzero initial probabilities q_i, are equivalent as probability spaces. The sets of all paths (Ω) are equal and the same sets of paths have probability zero.*

4 Realization

4.1 The Concept of Realization

The aim is to put regime swiching models, as introduced in the last chapter, into practice using an ordinary laptop. It should be possible to price options with a maturity from 1 day to 1 year or even longer with sufficient accuracy in one single model. The problem is the required size of the tree - the numbers of nodes and edges. In the strictly recombining classical CRR model the number of nodes just grows by one with every time step. This realistically cannot be maintained with the complexer regime switching models. But if the models do not make multiple use of the same $\widetilde{S}$-values, it inevitably leads to numerically untreatable exponential growth of the number of nodes.

4.1.1 The Lattice

The most important step to get numerically calculable trees is the construction of a lattice that contains all combinations of t_i and $\widetilde{S}_{t_i}$ occuring in the tree. By a *lattice* (or *grid*) we mean a $\mathbb{Z}$-module of rank 2 in $\mathbb{R}^2$ - one component for time, one for $\log \widetilde{S}_{t_i}$ (the regimes will be treated separately below).

To be precise, not the pairs $\left(t_i, \widetilde{S}_{t_i}\right)$ will be points of the lattice, but the pairs $\left(t_i, \log \widetilde{S}_{t_i} - \log S_0\right)$. A tree, for which such a lattice exists, might be called *semirecombining*. Note that a semirecombining tree need not be numerically calculable. ‘Semirecombining’ is more a necessary than a sufficient condition.

A suitable lattice can be constructed as follows: First a minimal time step Δ_0 as on page 14 has to be fixed. Next, a *minimal volatility* $\sigma_0 > 0$ also has to be fixed. The set of points $\left(i\,\Delta_0,\, j\,\sigma_0\sqrt{\Delta_0}\right)$ with integral i, j then form a lattice. In all examples below the values of Δ_0 and σ_0 are

$$\Delta_0 = \frac{1}{3285} = \frac{1}{9}\,day \quad \text{and } \sigma_0 = 0.01 \tag{7}$$

These values turned out to be small enough to calculate the prices of short running options with sufficient accuracy, as well as big enough to keep trees for long running options numerically treatable (at least if some accompanying measures are taken (cf below)).

4.1.2 Implementation of the Regimes

Of course, the requirement to stay within the lattice, imposes some conditions on the parameters of the regimes. For those of diffusion type (3.2.1 and 3.2.2) there are positive numbers σ_u and σ_d such that the price $\widetilde{S}_{t_i} = S_0\, e^{j\sigma_0\sqrt{\Delta_0}}$ in t_i becomes one of the values

$$S_0 \,\exp\left(j\,\sigma_0\sqrt{\Delta_0} + \sigma_u\sqrt{\Delta_{t_i}}\right) \text{ or } S_0 \,\exp\left(j\,\sigma_0\sqrt{\Delta_0} - \sigma_d\sqrt{\Delta_{t_i}}\right)$$

in t_{i+1}. The corresponding points lie on the lattice, if there are positive integral numbers k, k_u und k_d such that the following equations hold:

$$\Delta_{t_i} = k^2 \Delta_0 \qquad \sigma_u = k_u\,\sigma_0 \qquad \sigma_d = k_d\,\sigma_0$$

From now on, this is assumed to be true. As a consequence, not every positive integral multiple of Δ_0 can be the length of a time step. Only squares of natural numbers are allowed. This is the reason for the “odd” choice of $\Delta_0 = \frac{1}{9} day$ above. To refine the models further, the candidates for the next steps are $\Delta_0 = \frac{1}{16}\, day$ and then $\Delta_0 = \frac{1}{25}\, day$.

The restriction to square number multiples of Δ_0 can be coped with, as at any time it is possible to reduce the length of the next time steps - down to Δ_0, if required. Any multiple of Δ_0 can be made a point in time of a model.

Next to the jump ramifications of section 3.2.3. These are characterized by a jump factor jp and a jump intensity λ, which need to be translated in multiples of $\sigma_0\sqrt{\Delta_0}$. If Δ_{t_i} has the smallest possible value $\Delta_{t_i} = \Delta_0$, then there must be positive integers k_{jp} and k_u so that

$$S_0 \exp\left((j + k_u)\,\sigma_0\sqrt{\Delta_0}\right) \text{ and } S_0 \exp\left((j - k_{jp})\,\sigma_0\sqrt{\Delta_0}\right)$$

are the possible values at t_{i+1} ($\widetilde{S}_{t_i} = S_0\, e^{j\sigma_0\sqrt{\Delta_0}}$ as above). The jump factors jp can only be of the form

$$jp = \exp\left(-k_{jp}\sigma_0\sqrt{\Delta_0}\right)$$

which is not very restrictive. Choosing then $k_u \geq 1$ determines the next step completely, because the martingale property must be fulfilled. As already pointed out in subsection 3.2.3, this means that the equation

$$q_u \exp\left(k_u\sigma_0\sqrt{\Delta_0}\right) + (1 - q_u)\, jp = 1$$

has to hold, and hence:

$$q_u = \frac{1 - jp}{\exp\left(k_u\sigma_0\sqrt{\Delta_0}\right) - jp}$$

Formula 4 then leads to intensity λ. Given the system parameters Δ_0 and σ_0, the smallest possible intensity is always obtained by $k_u = 1$.

Now, let's assume the time step has length $k^2\Delta_0$ with $k > 1$. For the step down the same factor jp has to be applied, whereas the upward step changes. It has to be a factor of the form $\exp\left(\kappa_u(k)\sigma_0\sqrt{\Delta_0}\right)$ ($\kappa_u(k) \in \mathbb{N}$) and the martingale equation must hold exactly. Usually, this does not allow to get the desired intensity λ exactly. By the first Taylor polynomial of $log(1 + x)$ and exp one can see that k^2k_u is a good approximation of the ideal $\kappa_u(k)$. As integrality is required, this value is the best one available and hence has to be taken. In this argumentation it is assumed that even for the greatest k, that do occur, the probability of more than one jump in an interval $[t_i,\, t_{i+1}]$ is negligibly small.

4.1.3 Tree and Tree-cut

The results of the preceding section allow to represent a regime switching model consisting of finitely many BSM regimes, skew ramifications and jump ramifications as described in 3.2, such that every state of the model (tree) consists of one of the finitely many regimes and a point in a lattice. The possible parameters are subject to some restrictions, but these are not very tight.

But even then, a model may be too big to calculate, depending on the values of the parameters. For instance with $k_d = 500$ oder $k_{jp} = 2000$ - not uncommon values - it is clear that it is impossible to calculate the complete tree because the number of nodes increases too fast. Luckily, extreme changes of $\widetilde{S}$ in a short time (like many big jumps) are even more improbable. Such very rare ramifications thus have simply been cut off. This has been done without compensation. So by comparing the calculated total probability with the value it should have (1) and doing the same with the expected value of $\widetilde{S}_T$ (should be $\widetilde{S}_0$) the importance of the cut off part of the tree can be reliably estimated. The difference was negligible in all cases.

4.2 Examples

A series of examples is to follow to convey an impression of the "model kit". In all of these examples the same time pattern is used. One year is divided in 114 time steps, the smallest one has length $\Delta_0 = \frac{1}{9}\, day \approx 0.000304414$ (as already indicated above (equ. (7))). Δ_0 is the length of each of the first 18 time steps. The biggest time interval has a length of 9 days. Call option prices are calculated for the maturities 2 days, 1 week (= 7 days), 1 month (30 days), 3 months (90.78 days), 6 months (182.33 days) and 1 year (365 days). For additional explanations see the relevant examples.

Remember also that $\sigma_0 = 0.01$ is the smallest volatility step and observe that no interest rates need to be specified as only discounted values of assets are considered. This, of course, is only true, if interest rates are considered as being nonstochastic.

4.2.1 Black-Scholes-Merton Model

We start with a model, where the wanted result is completely known and calculable by the Black-Scholes formula. $\sigma = 0.25$ is the chosen volatility of the single regime used, which of course is of CRR type (BSM type). So there is no regime switching at all, but nevertheless, the model is not a CRR model, because of the varying length of time steps. The resulting tree has 17,348 nodes and 33,880 edges and thus is a pretty small tree.

The following table shows the call prizes of the model for various strikes K and the maturities as introduced above. The strikes are not noted directly but by the correspondig forward moneyness FWM, which is defined by

$$FWM = \frac{S_0}{\widetilde{K}} = \frac{e^{r\,t_i}\,S_0}{K}$$

(r denotes the continuously compounded interest rate for the period from

t_0 to t_i, the maturity date of the option). The option prices are denoted in percentage of S_0. All values are rounded.

FWM	1.44	1.30	1.18	1.09	1.00	0.91	0.81
2 days	30.33	23.18	15.30	07.90	00.67	00.00	00.00
1 week	30.33	23.18	15.30	07.91	01.32	00.00	00.00
1 month	30.33	23.18	15.32	08.33	02.79	00.28	00.00
3 months	30.34	23.24	15.78	09.73	04.91	01.59	00.25
6 months	30.44	23.64	16.81	11.44	06.99	03.34	01.11
1 year	31.01	24.81	18.78	14.02	09.90	06.07	03.08

The next table shows the differences between these prices and the prices computed by the Black-Scholes formula (the rounding took place after calculating the differences).

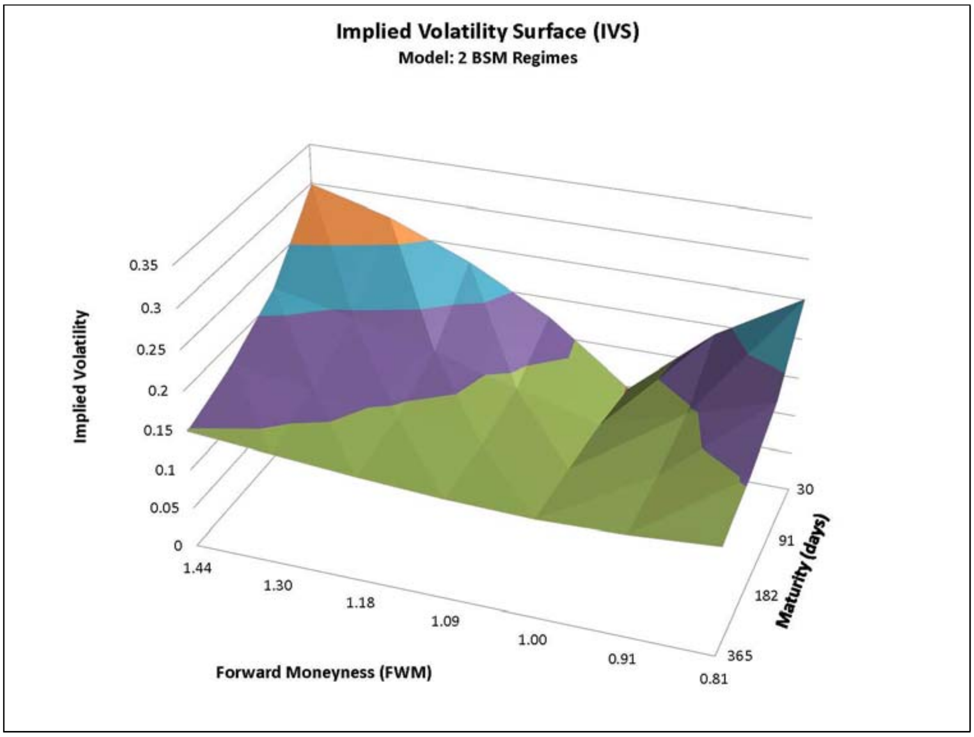
FWM	1.44	1.30	1.18	1.09	1.00	0.91	0.81
2 days	00.00	00.00	00.00	00.00	00.00	00.00	00.00
1 week	00.00	00.00	00.00	00.00	00.01	00.00	00.00
1 month	00.00	00.00	00.00	00.00	00.01	00.00	00.00
3 months	00.00	00.00	00.00	00.00	00.01	00.00	-00.01
6 months	-00.01	-00.01	00.00	00.01	00.02	00.01	00.00
1 year	00.00	00.01	00.01	00.01	00.01	00.01	00.00

Obviously - and not surprisingly - the short-running options benefitted from the small value of Δ_0, whereas the accuracy of the prices of the long running options is not quite as good, but still acceptable. So the longer time steps do not lead to bigger imprecisions, if the time steps in the near future are small - at least in this example.

This model is one of the very few skew tree models where the process of $\left(\widetilde{S}_t\right)$ is a Markov Chain. The increments of $\widetilde{S}_t$ and $\log\left(\widetilde{S}_t\right)$ of the time steps of the model therefore are independent and by the *characterisitc function* it can be shown that for every $t > 0$ $\log\left(\widetilde{S}_t\right)$ converges to a normal distribution for every sequence of such models with $\Delta_t \to 0$. This is even true for variable time steps, as long as the maximal time step converges to zero.

4.2.2 Multiple BSM Regimes

The following figure shows the Implied Volatility Surface (IVS) of a model with two regimes of BSM type. The volatilities are 5% (regime Z_1) and 40% (Z_2) (i.e. $k_u = k_d = 5$ resp. $= 40$).

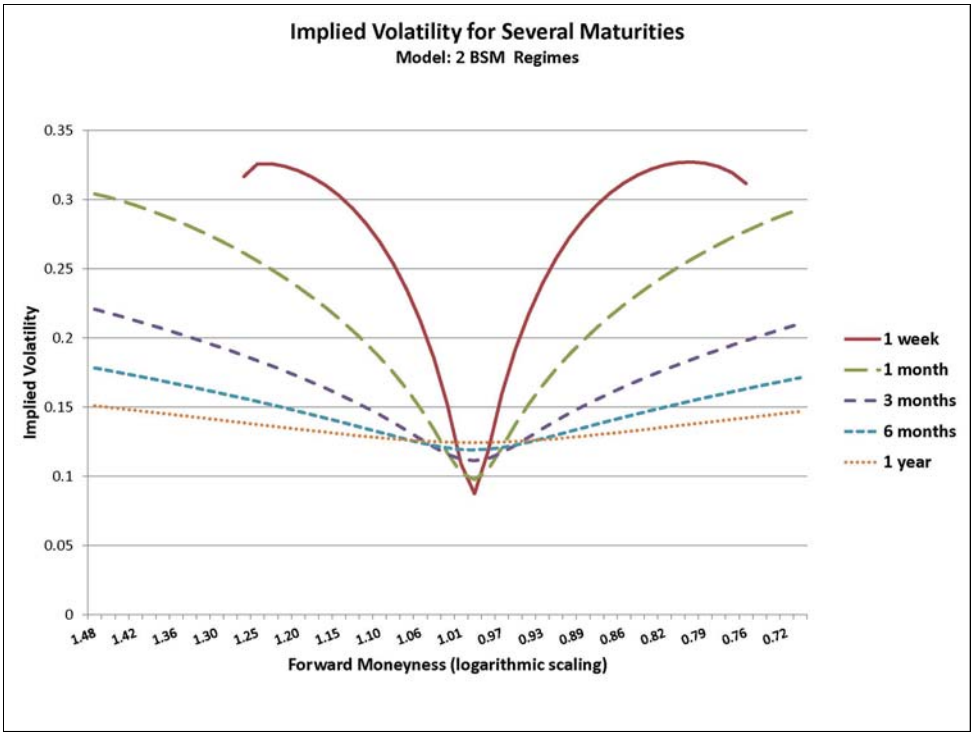


The chosen transition matrix

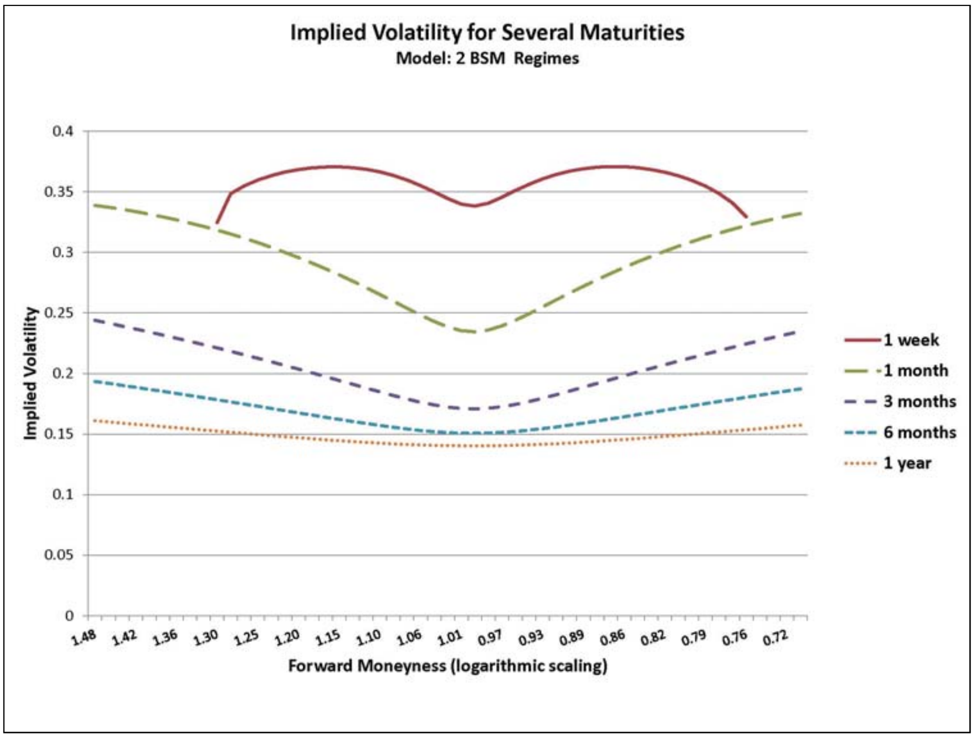
$$A = \begin{pmatrix} 0.999 & 0.001 \\ 0.01 & 0.99 \end{pmatrix}$$

has diagonal elements close to one, which is typical fo all our models. Regime switches are always possible, but they are also always more the exception than the rule.

The matrix A is vitally important for the shape of the IVS, as this matrix determines the equilibrium distribution of the regimes, which in this case was also taken as initial distribution (a steady-state model). The distribution in the example is $(0.909090..., 0.0909090...)$. This means that on the long run, more than 90% of the time the regime with the low volatility 5% will be active. A consequence of this is the comparatively low volatility of long-running options.



Smiles of a 2 BSM regimes steady-state model

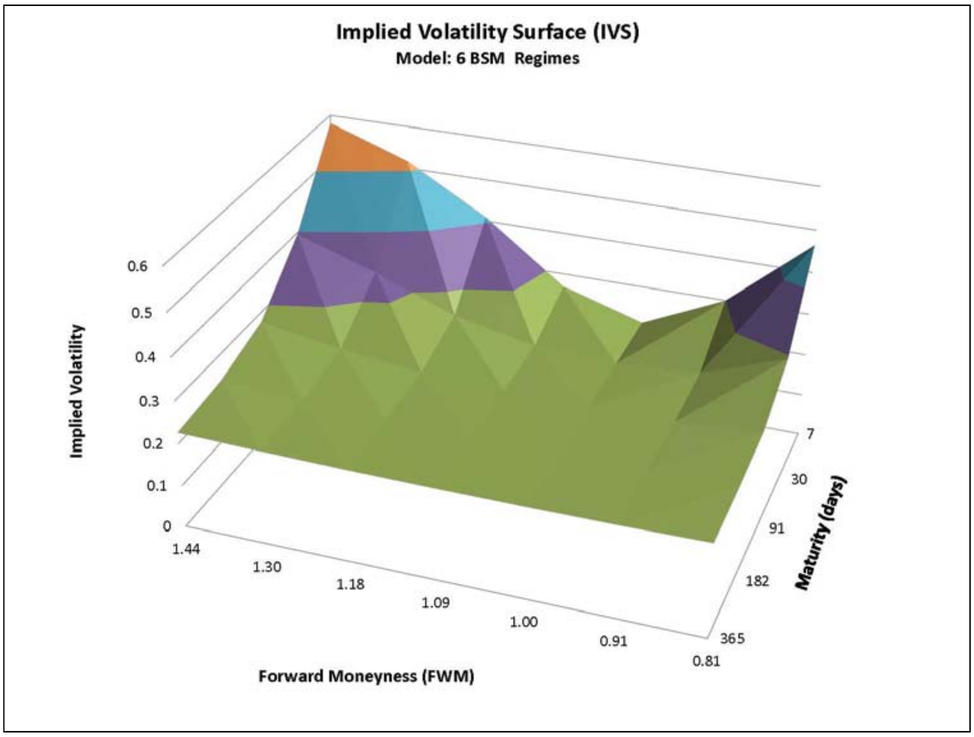


Smiles of a general augmented 2 BSM regimes model

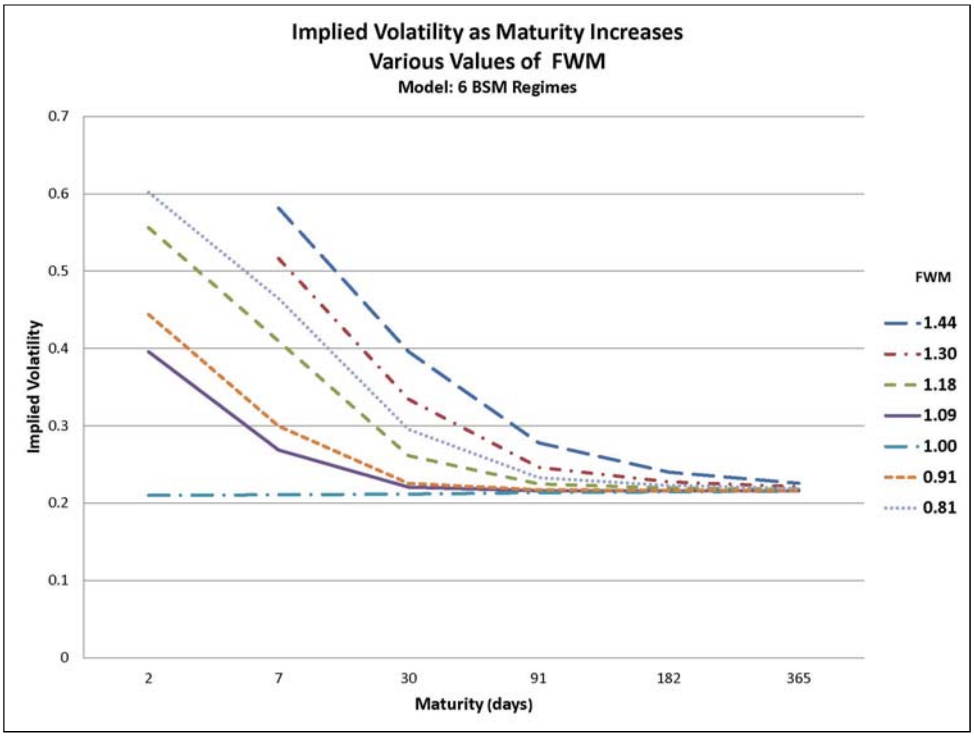
Some more general or at least typical properties: The higher the steady-state probability of the high-volatility regime Z_2, the higher the implied volatility of long-running options. A typical property of this section's models, which can be seen in the figure on page 22, is the apparent symmetric smile. This can be seen even better if one looks at the upper figure on page 23, which shows the implied volatity curves (smiles) of several maturities.

As to general augmented models, the initial distribution has great influence on the prices of short-running options. For instance, if in the example above $(0, 1)$ is taken as initial distribution (i.e. regime Z_2 rules), one obtains the lower diagram on page 23.

Some technical data: The underlying tree of the steady-state model has 450,852 vertices and 1,762,640 edges. The other model has similar values, but they are not exactly the same, because the tree-cut is a little bit different. In both models, after 1 year 99.999% of the total probability have escaped from tree-cut and also more than 99.999% of the expected value. So the resulting values (especially option prices as expected values) are very close to those of the corresponding ideal tree without tree-cut.
The next two figures belong to a model consisting of 6 BSM regimes - the maximum number of regimes in the study. In spite of the many regimes the



The implied volatility surface of a model made of 6 BSM regimes



model turned out to be computable on a laptop built in 2012 (but it was close to the limit).

One might be tempted to use tree models with many regimes in order to approximate models with continuoulsy varying stochastic volatility. The volatilities used in the model are 5, 10, 15, 20, 25 and 80 percent. The transition matrix is

$$A = \begin{pmatrix} 0.99 & 0.009 & 0.001 & 0 & 0 & 0 \\ 0.0005 & 0.999 & 0.0005 & 0 & 0 & 0 \\ 0 & 0.00009 & 0.9999 & 0.00001 & 0 & 0 \\ 0 & 0 & 0.000001 & 0.9999 & 0.00005 & 0.000049 \\ 0 & 0 & 0 & 0.0005 & 0.999 & 0.0005 \\ 0 & 0 & 0 & 0 & 0.01 & 0.99 \end{pmatrix}$$

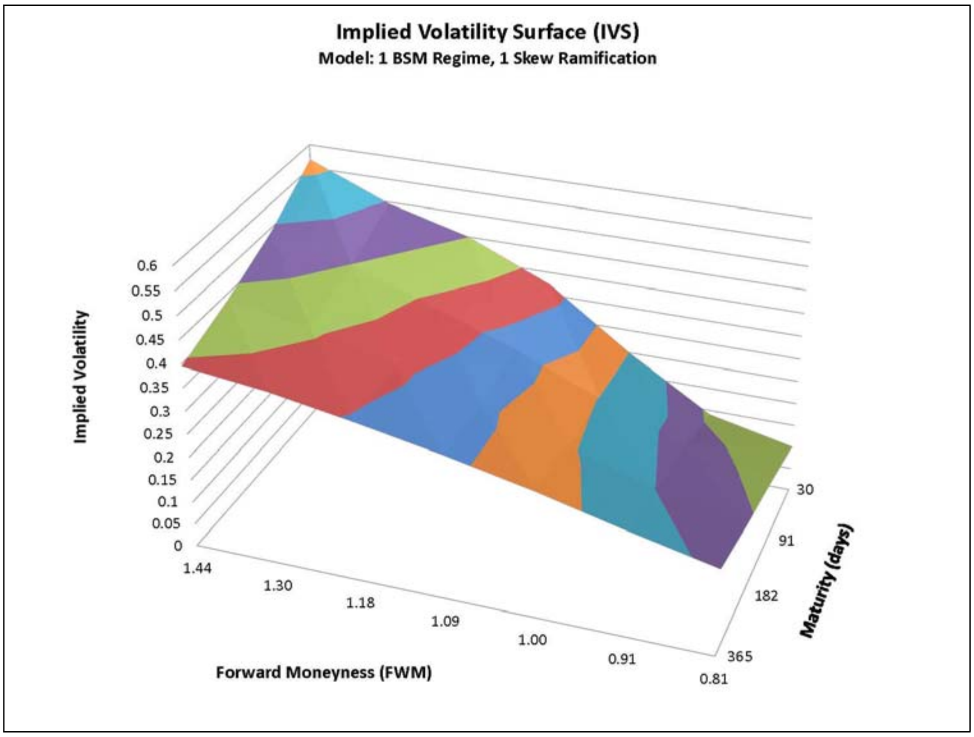
and the equilibrium distribution is used as initial distribution.

The symmetric smile is very much pronounced for short running options, but fades away quickly and is almost gone after 1 year. Then the implied volatility is only slightly greater than 0.2. This is to a great deal a consequence of the fact that the stable probability of the 20%-regime is bigger than 0.75. The development of the smile can also be read from the figure on this page. Some technical data of this model: $2,629,736$ vertices; $30,108,074$

edges; total probability after 1 year and after tree-cut: more than 0.9999; total expected value: nearly 0.999, 999.

4.2.3 BSM Regime Combined with Skew Ramification and Combined with Jump Ramification

Although quite a lot of volatility surfaces can be generated by the models described in the last subsection, those typically found in connection with stocks or stock indices are not among them. This is because in these models all tree ramifications are (close to) symmetric with respect to $\widetilde{S}$, which leads to symmetric distributions and symmetric smiles. But smiles of stocks and stock indices are to a great extent asymmetric. Typically, their implied volatility surfaces have the shape of a conic hill with summit at the point with maximal forward moneyness and shortest maturity. From this summit, the hill decreases in all directions more or less steeply, and more or less uniformly, into a more or less wide plain. The two model types of this subsection have no problem to generate volatility surfaces that at least principally have such a shape. Here is an example:



The model that led to this IVS is a steady-state model consisting of a BSM regime Z_1 ($\sigma = 0.12$) and a skew ramification Z_2 ($\sigma_u = 0.05$ and $\sigma_d = 5.00$).

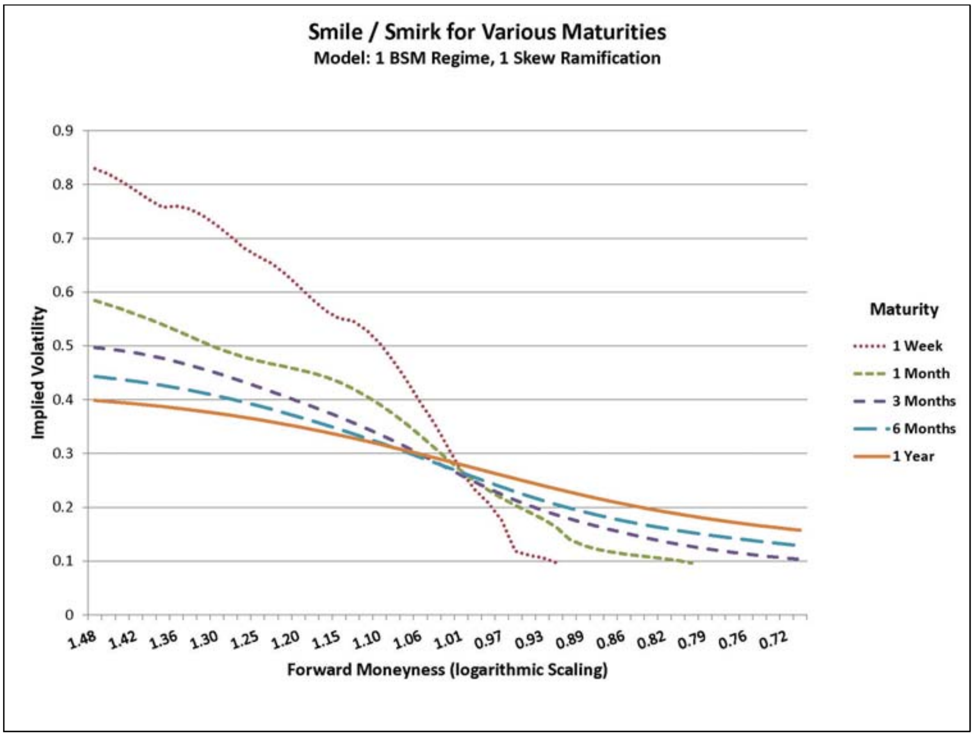
The large value of σ_d represents a crash scenario, in which the asset loses half of its value in one week (3/4 in one month), if this scenario is permanently active. The transition matrix used is

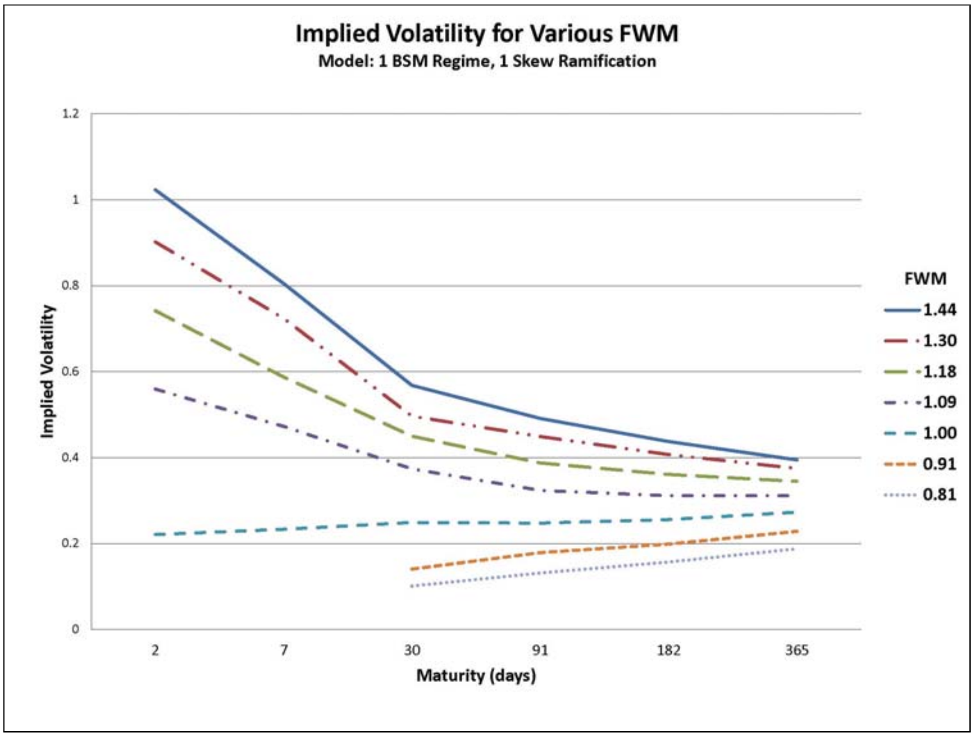
$$A = \begin{pmatrix} 0.995 & 0.005 \\ 0.01 & 0.99 \end{pmatrix}$$

which means that the stable probability of Z_1 is 2/3 and that the system in the long run will be in the horrible state of a move according to σ_d in 0.4% of the time - or about 1.5 days a year.

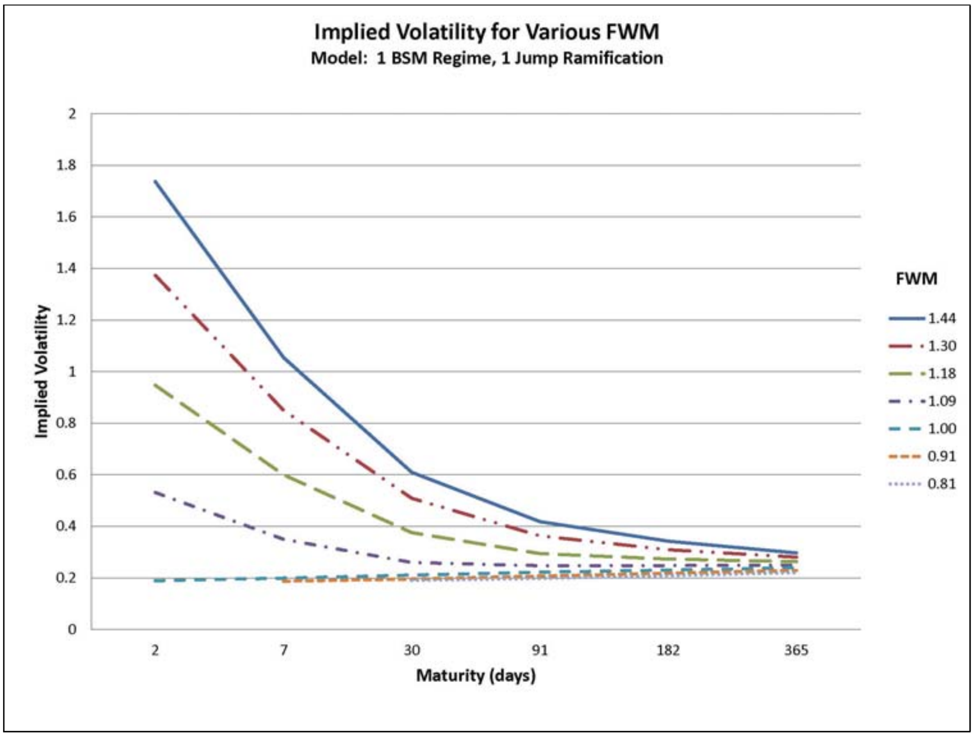
Because of the large σ_d-value of Z_2 the tree is growing fast, but varying time steps and tree-cuts maintain numerical computability. 0.03% total probability and 0.0003% expected value of $\widetilde{S}$ are cut off after one year. 2.2 million remaining nodes and 8.7 million remaining lines can be counted in the final tree (horizon 1 year).

The next two figures show smiles / smirks of the model and the volatility of maturities from 2 days to 1 year for various moneynesses.



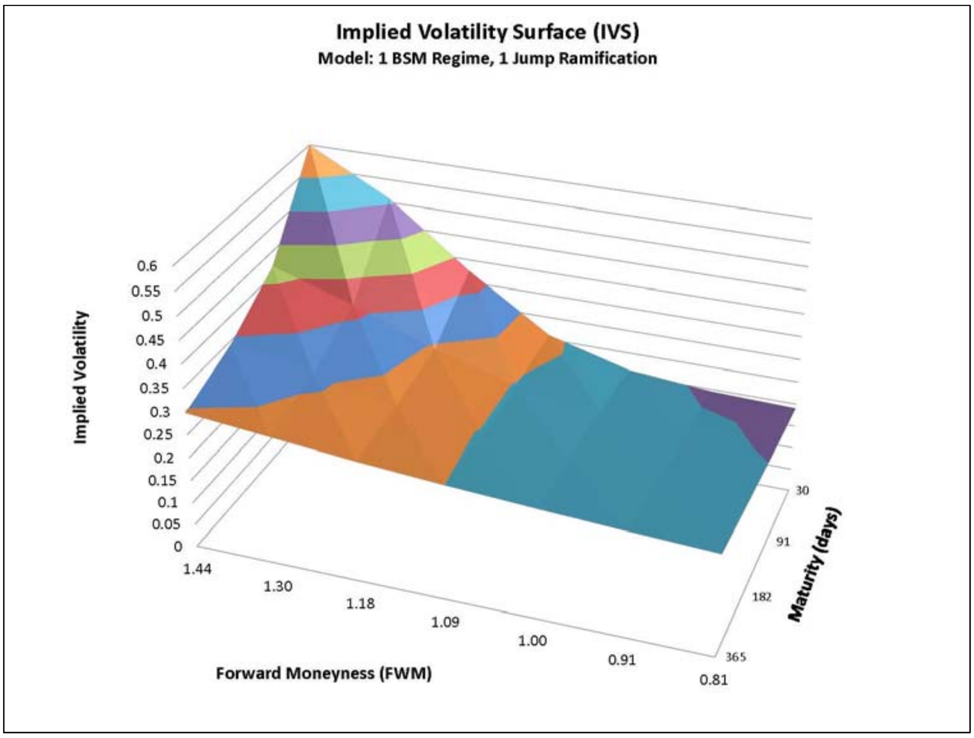


Even the 1-year options have a considerable smirk - the implied volatility reaches from below 0.2 to 0.4. This is quite different to the following picture:

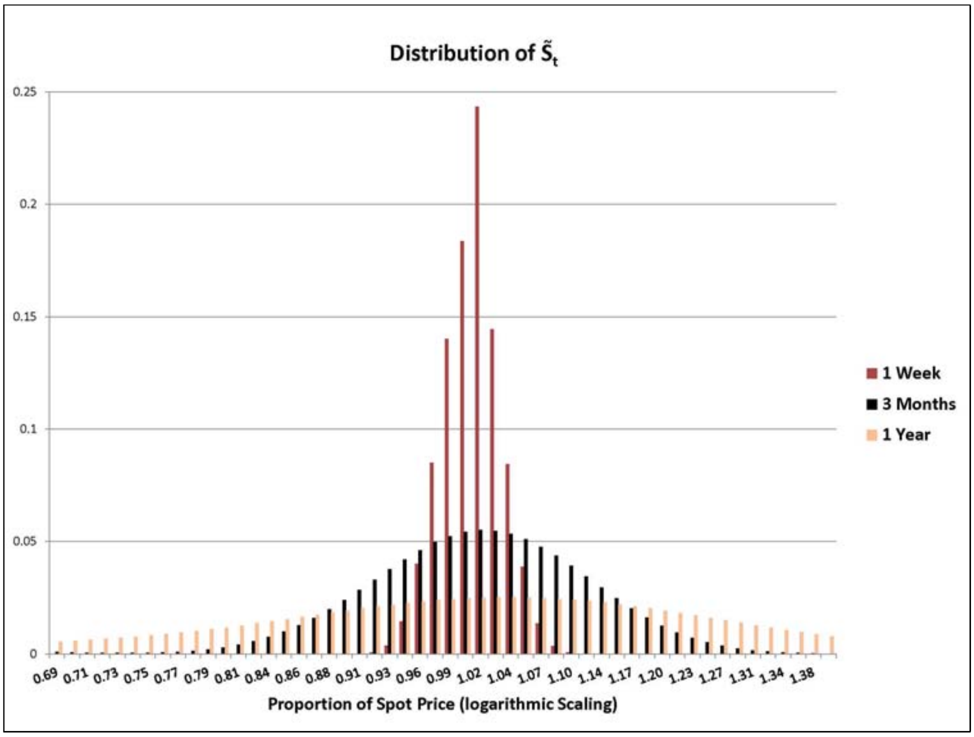


For a maturity of 1 year all implied volatilities are between 0.2 and 0.3, even though 2 days running options show a wide range from 0.2 to 1.8. These implied volatilities belong to a combination of a BSM regime ($\sigma = 0.2$) and a jump ramification ($k_{jp} = 3000$ and $k_u = 1$). So it is no surprise the model shows some properties that typically belong to jump diffusion models (cf [Cont Tankov 2004] and [Merton 1976] - the classical model in this model group). The event of a downward jump in this model means a loss of value of 40%. On average, this happens once in 7.8 years (risk neutral expected value), as can be derived from the parameter values.

The regime switching probabilities of the model are 0.01 (jump ramification to BSM regime) resp. 0.001 (BSM regime to jump ramification). The initial distribution is set to the stable distribution, which means the BSM regime has probability 0.909 (rounded). Within the jump ramification regime the event of a jump has intensity $\lambda \approx 1.4$. The IVS of this model looks as follows:



The implied volatility surfaces of this subsection fit to distributions of the underlying that are skewed left and steep to the right. For the current model this looks as follows (3 dates):



A combination of a BSM model and a jump ramification - distributions of $\widetilde{S}_t$

Especially for $t = 1\,week$ the skewness is well visible, whereas for 1 year it can hardly be recognized at all.

Just as with the preceding model, one single time step can result in big changes of $\widetilde{S}_t$ ($\rightarrow k_{jp}$), so an eye has to be kept on the technical data of the tree. Reassuringly the order of magnitude is the same (2.4 mio. nodes, 9.5 mio. edges) and the exhaustion after tree-cutting even is a little bit better (probability 99.99%, expected value 99.999%).

4.3 Summary and Prospects

Various combinations of BSM regimes, skew ramifications and jump ramifications have been realized. With the help of the ideas of chapter 4.1 tree models could be generated that were small enough to run on standard laptops and big enough to put the chosen parameter sets into effect. At least, no indication was found that the tools described in section 4.1 (*lattice*, *varying time steps* and *tree-cut)* generate unpredictable effects, if used appropriately (but see remark 16 on page 67). Some of the realised models have extreme parameter values and would not be executable without these tools or something equivalent.

An overview of the results:

A fairly small tree is sufficient to obtain good approximations of the Black-Scholes-Merton model for a wide range of maturities simultaneously.

Combinations of BSM regimes lead to models with smiles. The smiles can be distinctive, but are always symmetric.

The introduction of *skew ramifications* and *jump ramifications* makes it possible to generate skew implied volatility surfaces. The simplest of these models - combinations of a BSM regime and either a skew ramification or a jump ramification were investigated more closely.

No other investigated 2-regime model makes it so easy to generate extremely pronounced smirks for short-running options than the combination of a BSM regime and a jump ramification. But these smirks typically seem to fade away quite fast as maturity increases. This is verisimilar, as a loss of a certain percentage of value appears to be the more drastic, the shorter the time interval is in which it happens. This property of a fast smile depletion can also be observed with classical jump diffusion models like the Merton model [Merton 1976]. As a consequence, the field of application of these models appears to be limited. But the other model type - a combination of a BSM regime and a (strongly) skew ramification - is equally parsimonious (only 2 regimes) and can also generate asymmetric smiles / smirks. These smirks appear to be more durable, if wanted. All in all, this approach seems to be able to generate a great variety of smiles / smirks in the frame of a pretty simple model.

So this model was chosen to be tested for the capability of representing real market data. The results can be found in the second part of this article.

Part II
A Two-Regimes Tree Model and DAX Options

5 Introduction to the Studies and the Model Type

Based on a positive impression of the capability of these rather simple models in early trials, the model type consisting of a BSM regime and a skew tree regime was chosen to be tested systematically on historic option prices. The second part of this paper covers the results of these investigations.

It was intended to consider more or less quiet phases as well as phases in which turbulences could be expected. To achieve this, two periods during 2016 and 2017 respectively.were chosen.The first period consists of the 3 months from May to July in 2016. Almost in the middle of this period was the 23rd of June - the day of the so-called Brexit referendum deciding whether or not Great Britain should leave the European Community EU. The second period was a period in April / May 2017, in which the first and the second ballot of the French presidential election 2017 took place.

The investigated options are Eurex DAX® call options. Thus, both periods contain political events, that were (are) likely to have influence on economics, but did not take place in Germany, the home country of the investigated stock index.

At first it was investigated, how good the model type could produce the observed implied volatility surfaces (IVS), i.e. how good the option market prices of each day can be approximated by a model of the chosen type. Then the dynamics of the prices and models were considered, which is of fundamental and great importance when it comes to hedging.

5.1 Required Parameters

To define a 2-regime tree model consisting of a BSM-type step (regime Z_1) and a skew ramification (Z_2), values have to be given to the following parameters (remember that we already fixed $\sigma_0 = 0.01$ (smallest volatility difference) and $\Delta_0 = 1/3285$ year (smallest time step) in the first part):

- k_1 - a positive integer determining the volatility of Z_1, i.e. $u_1 = e^{k_1\sigma_0\sqrt{\Delta_t}}$ the upward factor and $d_1 = 1/u_1$ the downward factor
- k_{2u}, k_{2d} - positive integers, analogously leading to upward and downward factor of Z_2
- $q_{1,2}$ and $q_{2,1}$ - the probabilities of a change from state Z_1 to Z_2 resp. Z_2 to Z_1 in a time interval of length $\Delta_t = \Delta_0$ (least possible time step); both probabilities have to be greater than 0 and less than 1, of course
- (q_1, q_2) - the distribution of the regimes Z_1, Z_2 at the start ($0 \leq q_1$, $q_2 \leq 1$, $q_1 + q_2 = 1$)

Then the transition matrix A and the equilibrium distribution (π_1, π_2) of the regimes are given by

$$A = \begin{pmatrix} 1 - q_{1,2} & q_{1,2} \\ q_{2,1} & 1 - q_{2,1} \end{pmatrix}$$

and

$$\pi_1 = \frac{q_{2,1}}{q_{1,2} + q_{2,1}} \qquad \pi_2 = \frac{q_{1,2}}{q_{1,2} + q_{2,1}} \tag{8}$$

Not all parameter combinations were taken into consideration. The upward factor of Z_2 was always set to $k_{2u} = 1$, the smallest possible value. This in combination with $k_{2d} >> 100$ underlines the meaning of Z_2 as skewness regime.

Special attention was paid to steady-state models, i.e. models with (q_1, q_2) $= (\pi_1, \pi_2)$. Within these models, not only (q_1, q_2) are determined by $q_{1,2}$ and $q_{2,1}$, but in many situations the influence of the transition matrix A on call option prices turned out to be mostly depending on the equilibrium distribution. So, to reduce the number of parameters, $q_{1,2}$ was always set to 0.1 for steady-state models. It seems that this arbitrary fixing does not reduce by much the ability of these models to reproduce observed implied volatility surfaces. However, looking back, a smaller value - something like 0.03 - 0.05 - might have been even better.

Summing up, the steady-state models taken into consideration are determined by three parameters: k_1, k_{2d} and $q_{2,1}$ (or $\pi_1 = q_{2,1}/\left(q_{2,1} + 0.1\right)$). In all situations, attempts were made to cope with these models. Where this could not be done in a satisfactory manner, general augmented models were taken into consideration. Then the fixing $q_{1,2} = 0.1$ was also no longer sustained, and $q_{1,2}$ and $q_{2,1}$ (or $q_{1,2}$ and π_1) were considered as independent parameters. This applies to the central days of the Brexit period and to a couple of days at the beginning and the end of the French election period in 2017.

Remark 6 *To completely specify a model, a few more things have to be described - most of all the time sheet (which is different for each trading day). This can be found in A.2.*

5.2 How Do the Parameters Work?

The influence of the parameter values on call option prices will be illustrated by ceteris paribus considerations, i.e. by varying one parameter of a typical parameter set and keeping the others fixed. But before doing so, it has to be asked, what the most significant graphical representation of the totality of call option prices of a model is. A popular graphical representation is given by the implied volatility surface (IVS) (see section 4.2 of the first part). Being a 3-dimensional picture, such a graphic provides excellent qualitative impressions, especially demonstrating the relation of a model to the

Black-Scholes-Merton model. But figures of IVS are not very helpful to illustrate minor quantitative differences between two similar volatility surfaces. Quantitative data can be read easier from line diagrams, i.e. graphs of functions of one variable. Moreover, in reality standard traded call options do not exist for each possible combination strike/maturity, but only for quite a lot of strikes and just a few maturities. This is especially the case with Eurex DAX® call options, the type of call options that have been studied.

So, instead of drawing an implied volatility surface, it is sometimes preferable to present a handful of cuts through this surface, each one illustrating the call option prices of a certain maturity. As a further step away from implied volatility surfaces, the diagrams do not show implied volatilities, but prices in a normed way: time values (in the forward format) expressed in percentage of the spot price of the underlying.

The following is the precise definition of the forward form FTV of the time value. On this occasion some basic assumptions and notations will also be described.

The underlying S is assumed to be a stock or a stock index without dividend payments. S_0 denotes the current value of S. $C(K,T)$ is the current value of a European call option on S with strike K and maturity T. The inner value IV of an option is usually defined as the value of the option when exercised immediately, so for a call option as above $IV = (S_0 - K)^+ = \max(S_0 - K,\, 0)$.

The forward form FIV of the inner value of a call option with maturity T is defined similarly, but still takes interest into account. It is the maximum of 0 and the current value of the forward contract $(S_T - K)$ with maturity T, i.e.

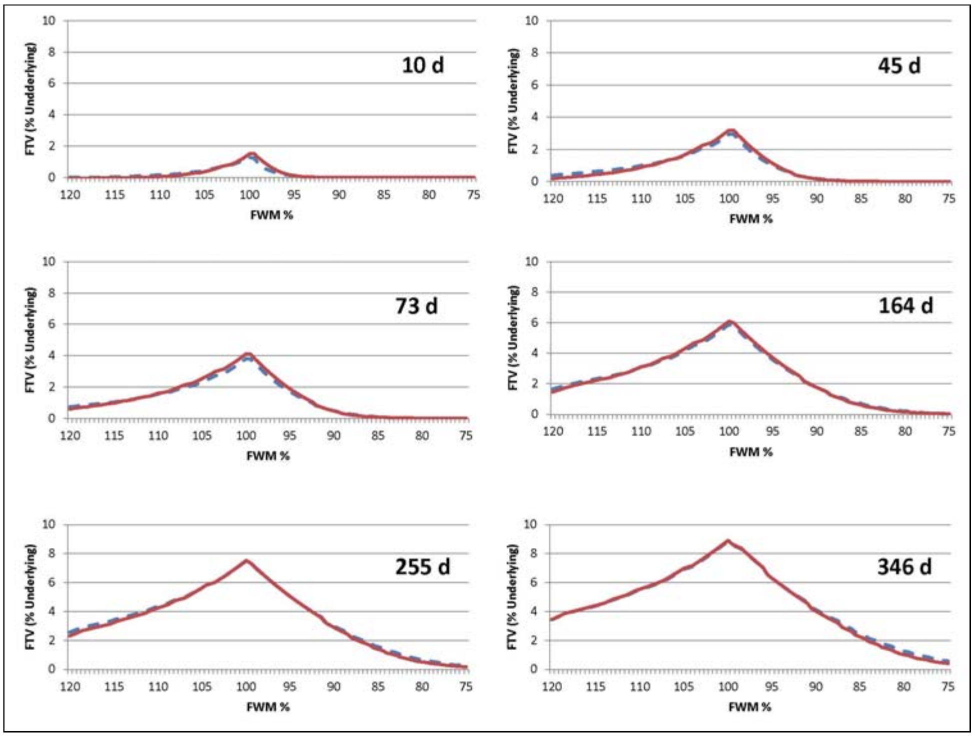
$$FIV\,(K,T) = \left(S_0 - \widetilde{K}\right)^+$$

The forward form $FTV\,(K,T)$ of the time value of a call option with strike K and maturity T is the difference

$$FTV\,(K,T) = C(K,T) - FIV\,(K,T)$$

This value expressed as percentage of S_0 is shown in the diagrams, i.e. the diagrams present

$$100\,FTV\,(K,T)\,/S_0$$

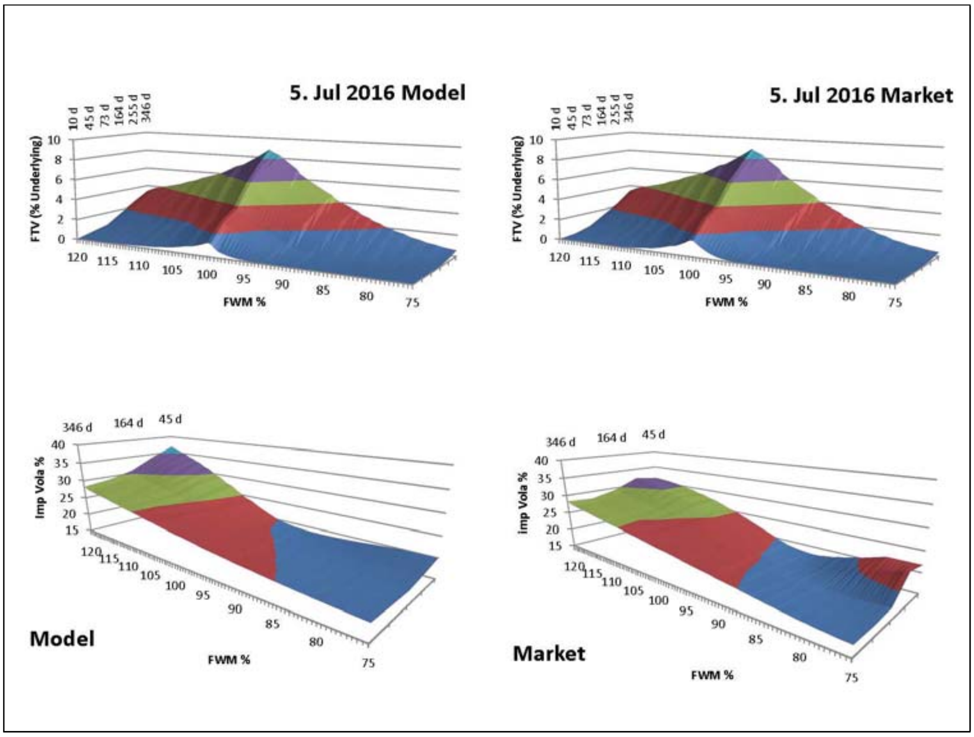


Option forward time values - market (solid line) and model (05.07.2016)

The Figure above shows an example. As in Part 1, the horizontal axis is scaled by the forward moneyness

$$FWM = \frac{S_0}{\widetilde{K}}$$

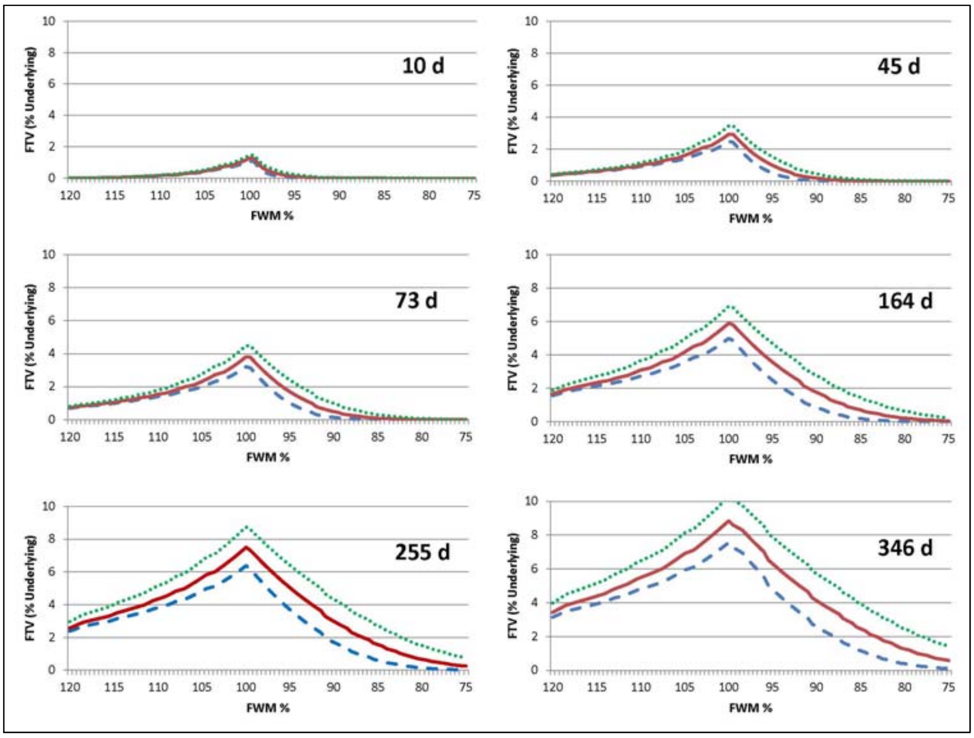
(and not by K). The solid lines show the FTV of the market prices of the Eurex DAX call options on the 05th of July 2016 with time to maturity between 10 days and 346 days. The dashed lines show the prices produced by one of the models we investigate in this part. It is a steady-state model with parameters $k_1 = 49$, $k_{2d} = 443$ and $q_{2,1} = 0.01093$. The diagrams show that the prices are close together, but there are some minor differences. In fact, we have found that this model provides the best fit to the prices of that day by a steady-state model. To measure the difference between two models or a model and market, the *norm distance* is introduced in the next chapter (section 6.2). Here the norm distance is 0.121647%, which is a good, but not excellent value.



3D charts: forward time value (FTV) and implied volatility surface (IVS)

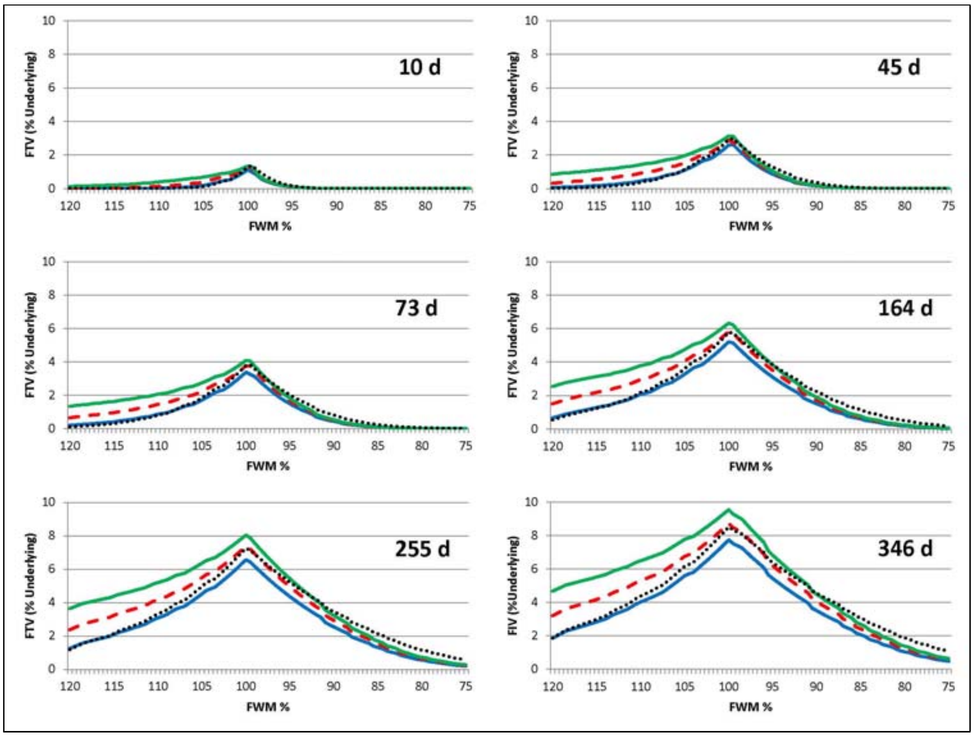
The figure on this page shows why it was not chosen to draw 3-dimensional diagrams of prices or implied volatilities. In the upper half 3-dimensional representations of model and market prices can be seen. Can you discover any differences? The impression given by the lower half is quite different. The implied volatility surfaces of the model and market can be seen there (without the 10d options, where some market prices did not allow to derive implied volatilities). Noteworthy differences seem to appear in the range of short running options. But these comparatively large differences of volatilities only lead to large relative price differences. The absolute price differences are small. On the other hand, small volatility differences can lead to large price differenes of long running options (does not occur in the diagram). So depending on the situation, a given difference of implied volatilities can sometimes mean large and sometimes mean small absolute price differences. A more direct impression of the price differences of two models or a model and a system of market prices is - naturally - given by the prices themselves. And as the inner value of a call option does not depend on the model, one can also compare the time values or forward time values. This has already been done on page 35.

Now we turn to the influence of the parameters and start with steady-state models. They have just three parameters: k_1, k_{2d} and $q_{2,1}$. The figure on this page shows the effect of a change in k_1, the sole parameter of the Black-Scholes-Merton regime Z_1. The values are $k_1 = 34$ (dashed line), 49 (solid) and 64 (dotted). The other parameters are $k_{2d} = 443$ and $q_{2,1} = 0.01093$. The picture looks very much the same as volatility variations in the BSM model do. Increasing the parameter leads to higher prices in a well balanced way. The rise increases with maturity and, given a maturity, is on average bigger for out-of-the-money options than for in-the-money options.



Varying the BSM parameter k_1

Someone who is familiar with the BSM model may wonder if the chosen values of k_1 aren't too great to be in a realistic range. But this is not the BSM model and "normal times" are not only represented by Z_1 alone. The upward step of Z_2 can belong to them as well. And this latter step has a much higher probability in a model with typical parameters than any of the two Z_1-steps (see section 5.3). As a consequence of this, the value of the BSM parameter k_1 is usually higher than the volatility of a BSM model.



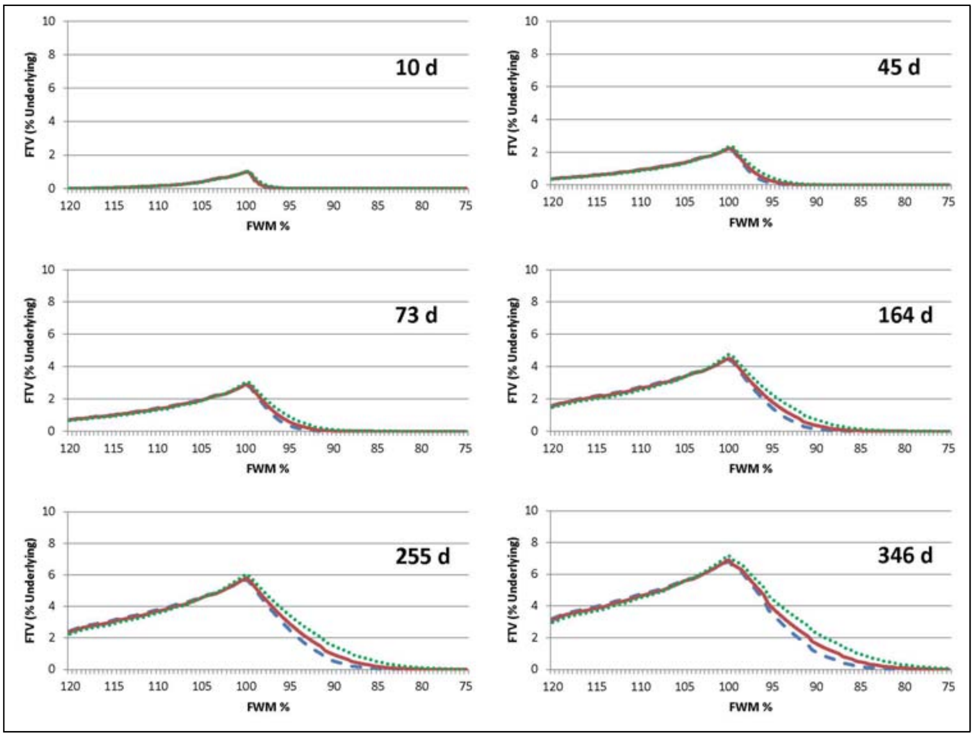
Varying the skewness parameter k_{2d}

The figure above shows why it is justified to call k_{2d} the skewness parameter. The parameter values $k_1 = 49$ and $q_{2,1} = 0.01093$ are completed by $k_{2d} = 200$ (lower solid line), 400 (dashed line) and 800 (upper solid line). You can see that raising k_{2d} does lift the option prices, but in a very asymmetrical way. Deep-in-the-money call options are becoming increasingly expensive, whereas deep-out-of-the-money options only slightly change their value.

The diagrams contain an additional, dotted line. It belongs to the BSM- (or CRR-)model with $\sigma = 0.22$. The comparison with this line shows that raising k_{2d} generates skewness. For deep-in-the-money call options the prices of the non skew BSM model are lowest, whereas deep-out-of the-money options are most expensive. This is true for all maturities.

For obvious reasons, k_{2d} might as well be called the *crash parameter*.

The impact of the third parameter $q_{2,1}$ on option prices to a high degree depends on the constellation of the other two parameters. $q_{2,1} = 0.005$, 0.01 and 0.02 combined with $k_1 = 49$ and $k_{2d} = 443$ leads to almost the same picture as is shown in the figure on page 37. This is becaues a high value of $q_{2,1}$ means the stable probability of Z_1 is also high, whereas the absolute stable probability of an upward step out of regime Z_2 decreases. These are the major effects on the option prices. The skewness component plays a less important role, but is affected too.



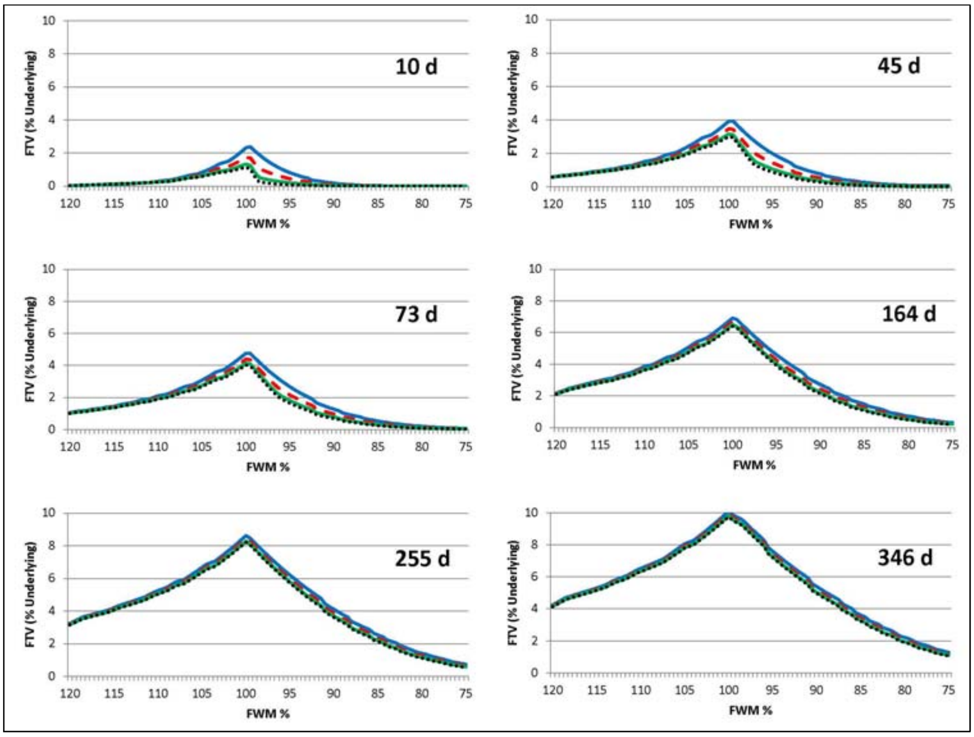
Varying the probability $q_{2,1}$ to leave Z_2

If one replaces $k_1 = 49$ by $k_1 = 25$ (and leaves everything else unchanged) the situation is quite different (the figure above; $q_{2,1} = 0.005$ (dashed line), 0.01 (solid) and 0.02 (dotted)). The option price level as a whole does not change very much when these changes of $q_{2,1}$ are made. But the skewness changes in a notable way. Raising $q_{2,1}$ leads to less skewness. Again, the reason is the loss of influence of Z_2 on the process of the value of the underlying, but this time it is the influence of the possible crash step according to k_{2d} that matters. Note that, in contrast to the figure on page 38, out-of-the-money options are more affected by the changes than in-the-money options.

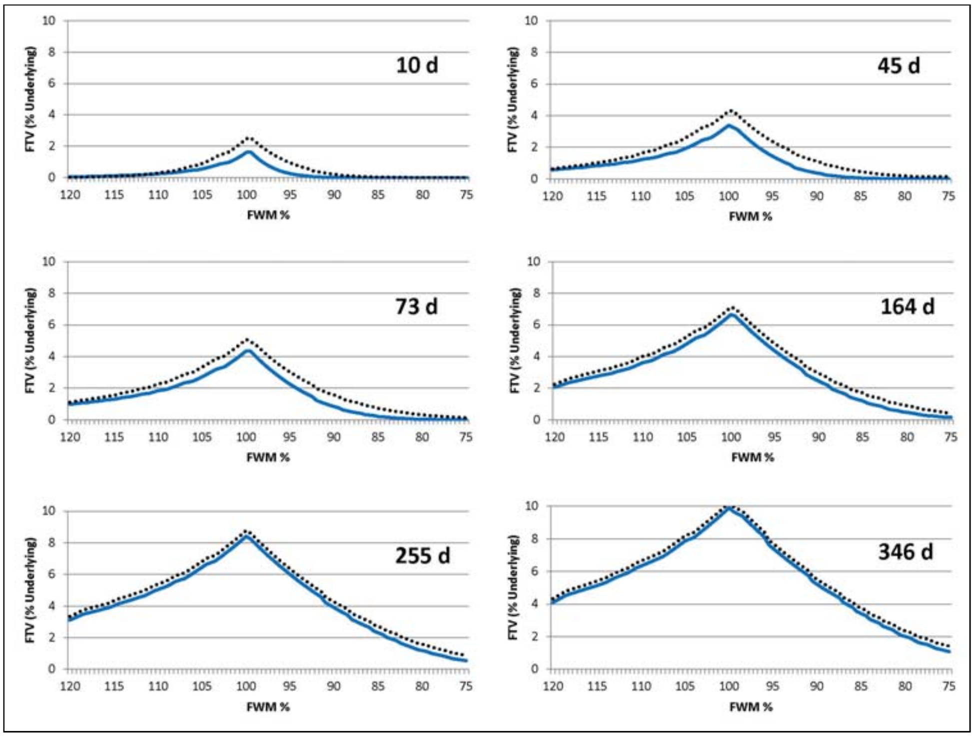
Summing up, there is a parameter for the volatility level (k_1), a parameter for the level of skewness (k_{2d}), and a parameter to do the fine-tuning ($q_{2,1}$). This makes it possible to generate quite a remarkable variety of implied volatility surfaces by models with stationary processes of $\log\left(\widetilde{S}_t\right)$ (as will be seen in sections 7.2 and 8.2).

But not every arbitrage free volatility surface can be obtained in this way. In all of the figures on the pages 37 to 39 the gaps between the lines get bigger, as time to maturity increases. This is a consequence of the fact, that the distribution of $\widetilde{S}_t$ spreads more and more, as t increases. If general augmented models are allowed, this restriction can be overcome in a certain (limited) way. For these models the system can start with an arbitrary Z_1/ Z_2-distribution and will finally converge to the stable distribution (8).

The figure below illustrates the impact of the initial distribution. The initial probabilities of Z_1 are 1 (upper solid line), 0.5 (dashed line), π_1 (dotted line), and 0 (lower solid line). The other parameters are $k_1 = 45$, $k_{2d} = 580$, $q_{1,2} = 0.01$ and $q_{2,1} = 0.002$. The stable probabilities are $\pi_1 = 1/6$ and $\pi_2 = 5/6$. This explains why the dotted line and the lower solid line are so close together. For the short running options ($10d$) the differences are biggest. The upper solid line almost shows time values of a BSM model, the lower one represents something like a pure skew model.



Varying the initial distribution



Varying the speed of convergence

The stable distribution only depends on the ratio $q_{2,1}/q_{1,2}$. The individual values of $q_{2,1}$ and $q_{1,2}$ determine the speed of the convergence. The smaller the values are, the less the probability is to change the regime in one step, and hence the slower the convergence is. If $q_{1,2} = 0.1$ and $q_{2,1} = 0.02$ had been taken, the convergence would be much faster. Only the $10d$-diagram would show a notable difference between the lines.

The figure on this page shows the speed of convergence by starting with Z_1 and $(q_{1,2}, q_{2,1}) = \lambda\,(0.1\,,\,0.02)$ with $\lambda = 1$ (solid line) and $\lambda = 0.06$. Everything else is as in the preceding figure. Notice that the lines get closer to each other only slowly. Also observe, that in each partial diagram the lower curve shows more skewness.

5.3 Different Ways to Look at the Model and Realistic Parameter Values

As already indicated, the range of realistic values of the BSM parameter k_1 is not the same as the range of $(100\cdot)$ the realistic volatility of a Black-Scholes-Merton model applied to the same situation. This can be seen by looking at the model in a different way which will be introduced now. The regimes can be "atomized" in a technical way: at each time t_i regime Z_1 can be regarded as the (disjoint) union of two regimes [3] Z_{1u} and Z_{1d}, where Z_{1u} is given, if Z_1 rules and the next step is an upward step. If Z_1 rules and the next step is a downward step, Z_{1d} is active. Analogously, Z_2 can be divided in Z_{2u} and Z_{2d}. Then each possible "situation" of the system (system = process of $\widetilde{S}$) belongs to exactly 1 of the 4 created new regimes and the process of these regimes Z_{1u}, Z_{1d}, Z_{2u} and Z_{2d} is a Markov chain with transition matrix ($t_i \rightarrow t_{i+1}$)

$$B = \begin{pmatrix} q_{1,1}\, q_{1u} & q_{1,1}\, q_{1d} & q_{1,2}\, q_{2u} & q_{1,2}\, q_{2d} \\ q_{1,1}\, q_{1u} & q_{1,1}\, q_{1d} & q_{1,2}\, q_{2u} & q_{1,2}\, q_{2d} \\ q_{2,1}\, q_{1u} & q_{2,1}\, q_{1d} & q_{2,2}\, q_{2u} & q_{2,2}\, q_{2d} \\ q_{2,1}\, q_{1u} & q_{2,1}\, q_{1d} & q_{2,2}\, q_{2u} & q_{2,2}\, q_{2d} \end{pmatrix}$$

where $(q_{k,l})$ is the transition matrix of the original regimes Z_1, Z_2 for the step $t_i \rightarrow t_{i+1}$ (depending on $t_{i+1} - t_i$) and q_{1u} ... q_{2d} are the respective upward / downward probabilities of these regimes for the same time step. The process is homogeneous if all time steps have the same length.

Whereas the process $\mathfrak{Z}$ of Z_1, Z_2 only determines the steps of $\widetilde{S}$ up to a probability distribution, the process $\mathfrak{Y}$ of Z_{1u}, Z_{1d}, Z_{2u} and Z_{2d} determines the development of $\widetilde{S}$ completely. There is a very close relation between $\mathfrak{Y}$ and the stochastic process $\mathfrak{X}$ of the combination $\left(\widetilde{S}, Z\right)$ $(Z = Z_1, Z_2)$. The correspondence between paths in both processes is illustrated in the following example. It is not a probability-conserving 1-1 correspondence in the strict sense, but it is very close, and it surely is justified to call the two processes *equivalent*. Thus, the process $\mathfrak{X}$ with infinitely many states bears the structure of a finite Markov chain (at least if all time steps have the same length).

Example 7 *The development*
$(S_0, Z_1) \rightarrow (S_0 d_1, Z_2) \rightarrow (S_0 d_1 u_2, Z_2) \rightarrow (S_0 d_1 u_2 d_2, Z_1) \rightarrow (S_0 d_1 u_2 d_2 u_1, .)$
is the union of two paths of length 4 in $\mathfrak{X}$, starting at (S_0, Z_1). It corresponds to the path $Z_{1d} \rightarrow Z_{2u} \rightarrow Z_{2d} \rightarrow Z_{1u}$ of length 3 in $\mathfrak{Y}$ in connection with the initial value S_0 of $\widetilde{S}$.
The probability of this path is $(q_{1,2}\, q_{2u})\,(q_{2,2}\, q_{2d})\,(q_{2,1}\, q_{1u})$ multiplied with the probability of a start in regime Z_{1d}. The probability of the 'development' in $\mathfrak{X}$ is $(q_{1d} q_{1,2})$ $(q_{2u} q_{2,2})$ $(q_{2d} q_{2,1})\, q_{1u}$ multiplied with the probability of a start in regime Z_1. The two probabilities are equal if

$$(\textit{probability of a start in } Z_{1d}) = q_{1d}\,(\textit{probability of a start in } Z_1)$$

[3] These are no regimes in the restrictive sense of 3.2, but they fulfil the requirement to guide the development of $\widetilde{S}$ over the next period (in an even stronger sense), which is the essential property of a regime and hence may be called regimes as well.

The discussion of the example shows how to get from a general augmented model $\mathfrak{X}$ with initial distribution (q_1, q_2) of the original regimes Z_1 and Z_2 to a corresponding model of the atomic type: one has to use the initial distribution $(q_1 q_{1u}, q_1 q_{1d}, q_2 q_{2u}, q_2 q_{2d}))$ of Z_{1u}, Z_{1d}, Z_{2u} and Z_{2d}.

Remark 8 *If an arbitrary initial distribution of Z_{1u}, ..., Z_{2d} is allowed in $\mathfrak{Y}$, this enlarges the set of possible models. It allows more general developments of the original regimes in the first time step. It has not been investigated, whether this is a useful extension of the model.*

The atomization of the model given by the regimes (in the extended sense of this section) Z_{1u} ... Z_{2d} and the process $\mathfrak{Y}$ allows to regroup the atoms according to special aspects. One question is, which regimes belong to "normal times" of stock exchange. As a first attempt one might think that Z_1, i.e. Z_{1u} and Z_{1d}, stand for a normal development and Z_2 stands for a crash scenario, but this is not true. If it were, the stable probability of Z_2 should be much smaller than the stable probability of Z_1. But the optimization processes of the next chapters almost always led to models where the opposite is true. By far the most steps of the best found models are ruled by Z_{2u}, i.e. are steps where $\widetilde{S}$ stays almost constant. For instance, a model with $q_{1,2} = 0.1$ and $q_{2,1} = 0.01093$, as illustrated in the preceding chapter, has steady-state probabilities $\pi_1 = 0.0109/0.1109 \approx 0.0983$ and $\pi_2 \approx 0.9017$. In general, π_2 is bigger than π_1 if and only if $q_{1,2} > q_{2,1}$. This is the case in all optimal models found below, but the difference is not always as big as here.

Back to the example above: If, in addition, k_{2d} is very large compared to k_{2u} (in the example $k_{2d} = 443$ and $k_{2u} = 1$), it is a consequence of the martingale property of the $\widetilde{S}$-process in regime Z_2, that on average in more than (say) 90% of the time Z_{2u} is the active regime. This indicates that Z_{2u} is a regime that almost completely belongs to "normal times". A step guided by Z_{2u} is only in exceptional cases a step of type "calm phase in rough times". This implies that BSM-parameter k_1 can be bigger than typical volatilities in BSM models. It is a different model with a different meaning of this parameter. In the studies k_1-values up to 60 turned out to be normal and in special situations values up to 100 occured.

After these observations one may be tempted to regroup the regimes in two regimes $U = Z_{1u} \cup Z_{1d} \cup Z_{2u}$ and Z_{2d}, following the idea that U represents normal times and Z_{2d} pure crash situations. But this does not produce a Markov chain[4]. A better system is given by $\mathfrak{W} = \left(\widetilde{S}_i, W_i\right)_{t_i}$ with

[4] To see this, let $k_{2u} = 1$ and $k_{1u} = 3$ (so that $u_2^3 = u_1$) and take a look at the paths

$W_i \in \{Z_1, Z_{2u}, Z_{2d}\}$. This system with three regimes is a Markov chain, because the first and the second line of matrix B are equal. The transition matrix C of $\{Z_1, Z_{2u}, Z_{2d}\}$ is obtained from B by dropping the first line and then substituting the first two columns by their sum:

$$C = \begin{pmatrix} q_{1,1} & q_{1,2}\, q_{2u} & q_{1,2}\, q_{2d} \\ q_{2,1} & q_{2,2}\, q_{2u} & q_{2,2}\, q_{2d} \\ q_{2,1} & q_{2,2}\, q_{2u} & q_{2,2}\, q_{2d} \end{pmatrix}$$

If not all time steps have equal length, this matrix of course has to be understood as time dependent.

The 3 regimes of $\mathfrak{W}$ have the following characteristics:

- Z_1 is a regime of BSM-type with comparatively high volatility
- Z_{2u} stands for quiet phases (which can be short), where $\widetilde{S}$ stays almost constant, but the market sentiment is positive
- Z_{2d} stands for crash scenarios, where $\widetilde{S}$ is on a straight way down

The process of $\widetilde{S}$ as a whole is a martingale, as is the process restricted to Z_1. Obviously, neither the process restricted to Z_{2u} nor the one restricted to Z_{2d} is a martingale.

Remember, that the process we are modelling is not the process of $\widetilde{S}$ under the real world probability measure $\mathbf{P}$, it is the process of $\widetilde{S}$ under the price determining equivalent martingale measure $\mathbf{Q}$. The model is incomplete, which means that the requirement of arbitrage free prices alone does not uniquely determine $\mathbf{Q}$ by $\mathbf{P}$. We do not address the question how, in the reality of trading, $\mathbf{Q}$ is selected from the set of all equivalent martingale measures of $\mathbf{P}$, but we call this selection the *market opinion* or *market opinion of risk neutrality.* So at least in a purely technical sense it may be understood that the probabilities of a suitable model represent the market opinion. Some interesting characteristic numbers may be derived from the model. For instance, it is obvious that the stable probability $\pi_{2d} = \pi_2 q_{2u}$ of Z_{2d} might be considered as the part of time the market expects the system to be in a crash state in the long run and $crash = exp\left(-k_{2d}\sigma_0\sqrt{\pi_{2d}}\right)$ is the factor by which the market expects the price of the underlying asset to be reduced by stock market crash times within an average year. But the studies did not support this interpretation throughout. The problem is that in some

$(S_0, U) \to (u_2 S_0, U) \to \left(u_2^2 S_0, U\right) \to \left(u_2^3 S_0, U\right)$ and $(S_0, U) \to (u_1 S_0, U) \to (S_0, U) \to (u_1 S_0, U)$. Both paths consist of three paths in the original process $\mathfrak{X}$, only differing by the last step in the second component. Now examine the next step.

situations (during the Brexit vote period) the models with the best fit to market prices had very high BSM parameters, so that crash-like scenarios within the model were not completely disjointed from Z_1 and thus could not simply be identified with Z_{2d}. Therefore, to quantify the market's fear of crash scenarios in a convincing way, a more complex approach seems to be required.

Anyway, k_{2d} is the true crash parameter. What are realistic values for k_{2d}? To decide this, an idea of how crash movements are in reality is helpful. Two probability measures on a finite probability space (the tree model) are equivalent iff the same elements (paths) have a probability greater than zero. So the possible movements under the price determining martingal measure are the same as those in (a suitable model for) reality. Only the probabilities of paths can be different und usually will be. Almost all best models in the chapters following have $100 < k_{2d} < 1000$. A day ($\Delta t = 1/365$) completely spent in the crash regime Z_{2d} leads to a loss of value of $1-\exp(-1\sqrt{1/365}) \approx$ 5.1% of $\widetilde{S}$ if $k_{2d} = 100$ and 23% if $k_{2d} = 500$. If $k_{2d} = 1,000$, the loss is 40.75%. These are big numbers, but the probability of such a day is extremely small (in reality and in the risk neutral world).

Remark 9 *The system $\mathfrak{W}$ has some apparent similarity to a model introduced by Rietz ([Rietz 1988]) on which Ang and Timmermann write in [Ang Timmermann 2012] (cf 2.4.1 Rare Events and Disasters):*
"In Rietz (1988), consumption follows a first-order Markov process with three regimes, two of which correspond to "normal" regimes and the third corresponds to a "crash" regime. The latter has zero probability of staying in the regime and equal probability of moving to the normal regimes."
In model $\mathfrak{W}$ the martingale property of Z_2 guarantees that Z_{2d} is reached from wherever with probability close to zero, if $k_{2d} >> k_{2u}$. Equal probability of where to go from Z_{2d} must not be given in $\mathfrak{W}$.

5.4 Aspects of Hedging

[5]In this article we do not treat hedging strategies in detail, but whatever a hedging strategy in a tree model may be, it is supposed to be able to replicate the price process of the derivative to be hedged by a self-financing trading strategy in some well defined assets called the *hedging instruments*. What this precisely means can be formulated in various ways. For a discounted tree

[5]Most of the results of this section do not only apply to the models of this chapter, but can easily be generalized to all skew tree models - at least under some additional conditions (like, for instance, the requirement that the transition matrix of regimes contains no zeroes).

model it can be expressed in the following way. Suppose a European option D with maturity T shall be hedged from today ($t = t_0 = 0$) to maturity date ($t = T$). The hedging instruments are A_1, ..., A_n, where each A_i equals the underlying S or is a derivative of S. For each node F (at time t_j) of the tree, that lies on a path from t_0 to T, the (discounted) values of D and of all the A_i must be uniquely determined[6]. In addition, the prices of the A_i must be apparent from market action.

Then a *hedging strategy* consists of a portfolio

$$P = -D + x_0 C + \sum_{i=1}^{n} x_i A_i \tag{9}$$

for each node N as above ($C =$cash(bond)), such that P has value zero in N and every adjacent node at time t_{j+1}. In other words, P helps the option writer to get from t_j to t_{j+1} without loss (and without gain). At t_{j+1} a node N' with a portfolio $P_{N'}$ is reached. The option writer can then transform P to $P_{N'}$ by buying/selling shares of the liquid assets A_i and is well equipped for the next period. This is a discrete version of delta-hedging as well as delta/gamma-hedging. Locally, at the nodes, it is clear what has to be done.

Example 10 *Binomial model (for example CRR model). Here, hedging is possible with cashbond and underlying asset alone, as it is a complete model. At* t_j *let* $\widetilde{S}$ *be the discounted value of the underlying with possible values* $u\widetilde{S}$ *and* $d\widetilde{S}$ *at* t_{j+1}*. Let* $\widetilde{D}$, $\widetilde{D}_u$ *and* $\widetilde{D}_d$ *be the corresponding discounted values of the option that is to be hedged. Then there are three equations to be fulfilled:*

$$\begin{array}{rcccccl} -\widetilde{D} & + & x_0 & + & x_1\widetilde{S} & = & 0 \\ -\widetilde{D}_u & + & x_0 & + & x_1 u\widetilde{S} & = & 0 \\ -\widetilde{D}_d & + & x_0 & + & x_1 d\widetilde{S} & = & 0 \end{array}$$

Because of the martingale condition for S *and* D*, the third equation is automatically true, if the other ones are fulfilled. Thus, two variables suffice.* x_1 *is the (discrete) delta of* D*.*

Notice that in the binomial model as well as in the general situation of formula (9) it is sufficient to consider the instantaneous data - as long as the prices are reliable.

Now, we turn to our special skew tree model, and at first consider nodes $N = \left(\widetilde{S}, Z_k, t_j\right)$ with a well defined regime $Z_k \in \{Z_1, Z_2\}$. This condition

[6] This is the reason why the original model trees are only able to price path-independent options and otherwise need to be extended accordingly.

only excludes the starting points of non-pure models. We assume that all considered options D and A_i are of European type, so that equation (1) can be applied. This allows to calculate option prices also for non liquid options, because the "hidden" regime Z_k is not invisible at all. It can be identified with the help of the prices of the liquid derivatives - at least in the model's world.

Let u_k and d_k be the upward- resp. downward-factor of Z_k. Then there are four adjacent nodes at t_{j+1}: $N_1 = \left(u_k\widetilde{S}, Z_1, t_{j+1}\right)$, $N_2 = \left(u_k\widetilde{S}, Z_2, t_{j+1}\right)$, $N_3 = \left(d_k\widetilde{S}, Z_1, t_{j+1}\right)$and $N_4 = \left(d_k\widetilde{S}, Z_2, t_{j+1}\right)$. This means, there are 5 equations (9) to be respected. Again, the number of hedging instruments needed can be reduced by the martingale conditions, but it is not possible to get along without derivatives as the considered model is incomplete.

An important fact is that the adjacent nodes at t_{j+1} come up in pairs of nodes that only differ by the reigning regime. (N_1, N_2) and (N_3, N_4) are the two pairs. Now take one of the pairs - say (N_1, N_2) - and consider the subtree of the model tree that consists of all nodes that can be reached from N_1 or N_2. This subtree has two nodes without predecessor: N_1 and N_2. Join these two nodes by an upstream augmentation step to get a general augmented model that starts at t_{j+1} (remember the augmentation step requires no time). Let q be the probability that the augmentation step leads to N_1.

Lemma 11 *If a portfolio P as in equation (9) satisfies this equation in N_1 and in N_2, then this equation also holds for P in the root node of the constructed general augmented model. This is true for every q ($0 \leq q \leq 1$).*

Proof. This is an immediate consequence of formula (6). ■

And an immediate consequence of the lemma is the following result, which - in a limited sense - allows a change of model within a hedging process:

Theorem 12 *At each step of a hedging strategy, an instantaneous change of augmentation leads to a correct hedging situation, i.e. the formulae (9) are still true under the modified probabilities.*

This means that if - according to a hedging strategy - an option writer builds up a portfolio P that satisfies the 5 equations in t_j (node N), (s)he is not only well prepared for the nodes N_1 ... N_4 in t_{j+1}. It is no disadvantage if the derivative prices then turn out to be the prices of a general augmented model for some q ($0 < q < 1$ instead of $q = 0, 1$). If D and the shares of the liquid derivatives can be bought/sold to prices in line with the market, the portfolio can be closed without loss and gain. If the hedge is to be continued

- i.e. D is not bought - a new hedging portfolio has to be created. This has to be done according to the model that belongs to q, and the situation is that of the starting point of a non-pure augmented model - the situation that has been excluded so far.

In this situation, it would be nice if it were possible to fill the artifical augmentation step with life, i.e. to get from a state with $\widetilde{S}$ known and regime insecure to a state with well-defined regime Z_k (and $\widetilde{S}$ and t unchanged). Three equations of type (9) have to be fulfilled, which is not very much, even though the (constant) underlying is not able to serve as a hedging instrument for this step. But the problem is that the market prices of options must explicate the corresponding changes and immediately afterwards the option holder must rearrange the hedging portfolio - in no time at all. To put it mildly, it is doubtful, whether this is possible.

What are the alternatives? One possibility is to generate a hedging portfolio that is well equipped for the challenges of the first two steps without any rearranging. This is possible, but the number of equations to be fulfilled inreases considerably from 3 to 11.

There is still some work to be done to find an optimal hedging strategy that minimizes the number of hedging instruments as well as the hedging error. The problem is not only restricted to the augmentation step. It occures as well in connection with "ordinary" steps, as a time step in the model can be a small step - much smaller than a trading day.

Remark 13 *Reaugmentation, i.e. replacing the value of q by another, is a change of model, but the models are closely related. The paths with positive probability are the same, but the probabilities are different. So the probability measures are equivalent. Under all of these measures the process of $\widetilde{S}$ is a martingale.*

6 Model and Market

6.1 Market Data

The market data studied in this research are prices of European DAX call options of Eurex over two periods in 2016 / 2017. DAX® is the blue chip index of Deutsche Börse AG, and Eurex Deutschland is Germany's leading derivatives exchange. It is a public company wholly owned by Deutsche Börse.

DAX consists of 30 major German companies traded at the Frankfurt Stock Exchange. A notable property of DAX is that it is a performance index.

Dividends of the involved companies are included in the index formula. DAX can be considered as a stock paying no dividend, which makes option pricing easyer.

Eurex provides DAX call and put options (ODAX) of European type. In every month there is exactly one final settlement day (maturity date). It is the third Friday, if this is an exchange day (otherwise the exchange day immediately before). Among the traded maturities are always the three nearest final settlement days and the nearest March, June, September and December final settlement days [7]. All investigated options have their final settlement day among these. For each trading day of the study 5 - 7 maturities were examined.

6.2 Norm Distance and Best BSM Model

The settlement prices of the call options of a trading day (market prices) define an implied volatility surface (IVS)[8], as do the model prices of the same options for each parameter set. The main task of this study was to find a parameter set that produces option prices (or IVSs) that are as close as possible to the prices (or implied volatilities) of the market. In order to do this, the first question is how to measure the distance between two sets of option prices. Should price differences be considered or volatlity differences? Is it adequate to consider absolute differences or is it better to use relative differences? Should any weights be introduced?

After the discussion of section 5.2 it should be no surprise that the decision was to compare prices (and not implied volatilities). More precisely, absolute (and not relative) price differences were considered, in order not to overestimate the influence of low-price options. Another argument is that the difference of two prices of a call option equals the difference of the forward time values inherent in the prices.

On this basis the least square method is used. Let $MA = (MA_1...MA_n)$ be the market prices of n options and let $MO = (MO_1, ..., MO_n)$ be their model prices. Then

$$d^2\left(MA, MO\right) = \frac{1}{n}\sum_{i=1}^{n}\left(MA_i - MO_i\right)^2 \tag{10}$$

is the mean quadratic distance of the two price vectors. In a first attempt this is the quantity that has to be minimized by an optimal model. But for each trading day there are only a couple of maturities that are considered.

[7] For further details see http://www.eurexchange.com

[8] Of course, this is only approximately true. There are only finitely many market prices.

Each maturity should have the same influence on the optimization process. This is only given by formula (10), if the number of options is the same for each maturity, which is not the case with Eurex options. So a weight factor is introduced. Suppose there are m maturities and n_j options of maturity j ($1 \leq j \leq m$). Let MA_j and MO_j be the vectors of market prices resp. model prices of options of maturity j. Then

$$ND^2(MA, MO) = \frac{1}{m} \sum_{j=1}^{m} d^2\left(MA_j, MO_j\right)$$

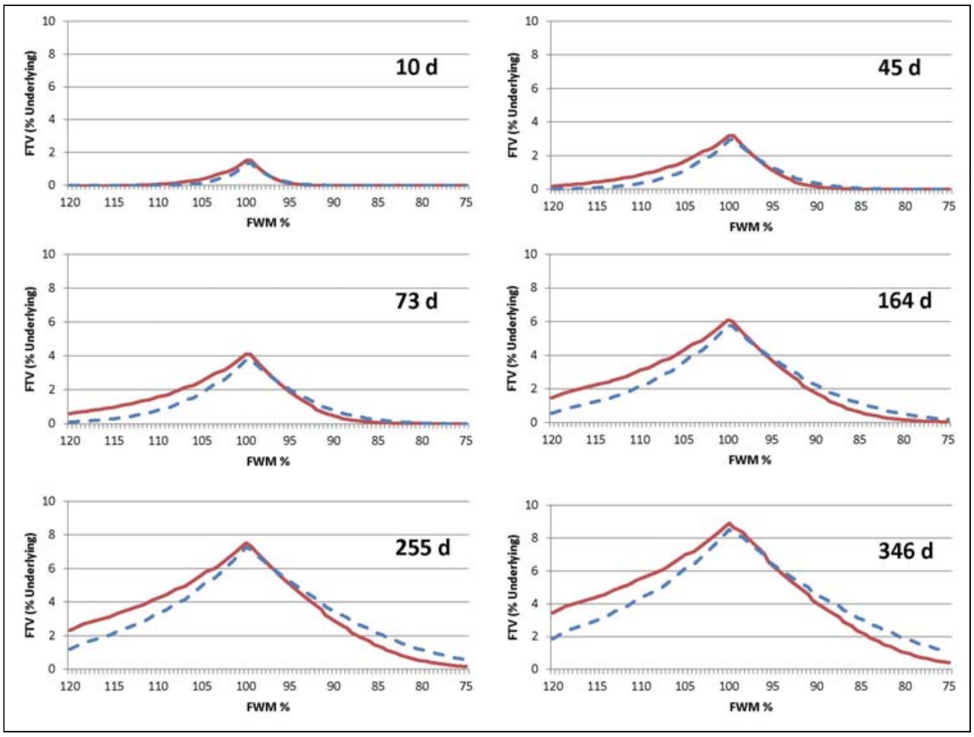
is a weighted sum of the squares of the price differences and this is the quantity that was minimized. Instead of doing this, one might as well minimize the square root. We call

$$ND(MA, MO) = \sqrt{ND^2(MA, MO)}$$

the *norm distance* of the market prices and the model prices. This quantity characterizes the average of the absolute values of the price differences better than ND^2. This has the same reason as in statistics: the standard deviation is more an average distance from the mean of a sample than the variance. Moreover, ND is a mathematical norm on $\mathbb{R}^n$ ($n = \sum n_j$), closely related to the Euclidean norm.

Just as in section 5.2, prices and related quantities as the norm distance will always be expressed as percentage of the underlying's price (in the studies daily closing price as stated in Eurex' daily online statistics). In the example at the beginning of section 5.2 the value of ND^2 is $0.014798\%^2$, so the norm distance between market prices and model prices is 0.12165%. This value is not based on all call option prices listed by Eurex. For instance options with very low or very high forward moneyness are not included. Details can be found in A.1.

Best BSM Model. The Black-Scholes-Merton model surely is no contender, if looking for a model that produces smiles and smirks. But it can serve as an easy-to-compute benchmark. Every model that claims to be able to deal with skewness must be able to generate prices that are significantly closer to market prices than those of each BSM model. The norm distance can be taken as a measure. For each trading day of the study, the BSM model was determined, which has a minimum norm distance from market prices. The search was restricted to volatilities $\sigma = k_{BSM}\,\sigma_0$ that are integral multiples of σ_0. The figure on the next page shows market prices (solid lines) and prices of the best BSM model ($k_{BSM} = 22$) of the example above. The norm distance is 0.51126% which is more than $4\times$ the norm distance between market prices and prices of the denoted skew tree model.



Market prices (solid line) and prices of best BSM model (05.07.2016)

6.3 Parameter Adjustment

For each trading day the task is to find a parameter set that minimizes the norm distance ND between the model prices and the settlement prices of that day. In principal, this can be done by searching local minima of ND (or ND^2) starting from various points and moving in directions where ND decreases, as long as this is possible. If this is done for a sufficiently large collection of starting points, chances are good that the determined set of local minima contains a global minimum or at least a parameter set with a norm distance close to the optimal value. Still the risk of considerably missing the global minimum cannot be eliminated completely in finite time - at least if no further structural assumptions are made on the set of market prices.

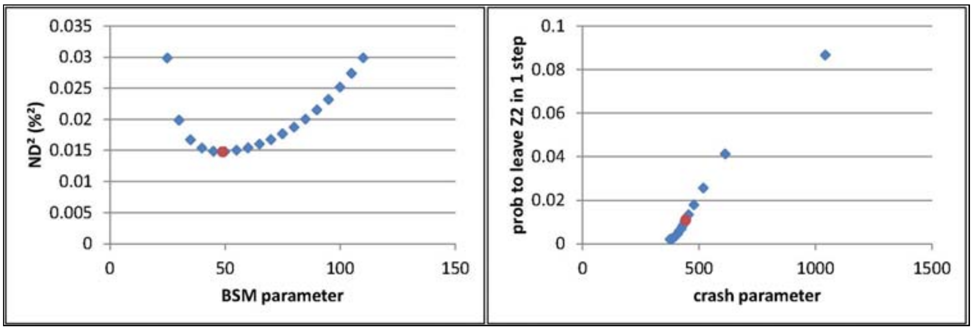
This risk can also not be excluded by the following more structured approach for every possible market data constellation, but in many cases the method turned out to be successful and helped to save time. Almost all of the most suitable steady-state models were determined the following way:

- First choose a grid of BSM parameters k_1 that covers the presumptive range for this parameter, for instance $k_1 \in \{15, 20, ..., 95, 100\}$

- Then find the best model for each of these values of k_1 (or at least get close to this). If the search is restricted to steady-state models (with in addition $q_{1,2} = 0.1$), this is an optimization problem with just 2 variables k_{2d} and $q_{2,1}$. In many cases (not all), the ND^2 values of these models form a nice convex curve showing a clear minimum. Moreover, k_{2d} and $q_{2,1}$ vary regularly (see the figure and the discussion below).

- Refine the grid near the ND^2-minimal point to get more precise values.

Why do the parameters typically behave as shown in the next figure? Arguments can be taken from the discussion in 5.2.



Best ND^2 value with given BSM parameter and the corresponding $q_{2,1}/k_{2d}-$combinations (05.07.2016)

Raising k_1 makes all call option prices increase. This has to be compensated at least partially by changes of the other parameters. Deepening $q_{2,1}$ lowers π_1 and hence the influence of regime Z_1 on option prices. Deepening k_{2d} also lowers call option prices. With only a few and marginal exceptions - which can also be due to the small inaccurancies that are inevitable in tree models - a grid of increasing k_1-values led to decreasing values of k_{2d} and $q_{2,1}$.

There is yet another argument that the optimal values of k_{2d} and $q_{2,1}$ change smoothly when the predetermined value of k_1 is varied. It is a consequence of the following result:

Lemma 14 *a) In a steady-state model with parameters k_1, k_{2d} and $q_{2,1}$ (and $k_{2u} = 1$, $q_{1,2} = 0.1$) and first time step of size Δt, the following identity holds for the expected value of the logarithm of the multiplicative first $\widetilde{S}$-step:*

$$\mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right) = \sigma_0\sqrt{\Delta_t}\left[\pi_1\left(q_1 k_1 - q_1`k_1\right) + \pi_2\left(q_2 - q_2`k_{2d}\right)\right]$$

Here q_i $(i = 1, 2)$ is the upward probability of regime Z_i, $q_i` = 1 - q_i$. π_i is defined as in equation (8).
b) If the first N time steps $(1 < N \in \mathbb{N})$ all have size Δt, then

$$\mathbf{E}_Q\left(\log\left(\widetilde{S}_N/\widetilde{S}_0\right)\right) = N\ \mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right) \tag{11}$$

Proof. a) Steady-state models start with the equilibrium distribution of the regimes. The rest follows directly from definition.
b) For each t_i the regimes are distributed according to the equilibrium distribution. This implies that for each i $(0 \leq i \leq N-1)$ the random variable $\log\left(\widetilde{S}_{i+1}/\widetilde{S}_i\right)$ has the same distribution and hence the same expected value as $\log\left(\widetilde{S}_1/\widetilde{S}_0\right)$. Now, by elementary properties of logarithm

$$\log\left(\widetilde{S}_N/\widetilde{S}_0\right) = \sum_{i=0}^{N-1} \log\left(\widetilde{S}_{i+1}/\widetilde{S}_i\right)$$

The summands are not independent, but that is no problem. The expected value of a sum of random variables is the sum of their expected values (if defined), even if they are not mutually independent. ■

Now, let Mo_1 and Mo_2 be two steady-state models that both provide good approximations to market prices of options of maturity $N\,\Delta t$. Then, under the price determining equivalent martingale measure, the probability distributions of $\widetilde{S}_N/\widetilde{S}_0$ have to be almost identical for both models because this distribution is determined by the totality of all prices of call options of this maturity. It follows that

$$\mathbf{E}_Q\left(\log\left(\widetilde{S}_N/\widetilde{S}_0\right)\right)_{Mo_1} \approx \mathbf{E}_Q\left(\log\left(\widetilde{S}_N/\widetilde{S}_0\right)\right)_{Mo_2}$$

Especially for N large, formula (11) forces $\mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right)_{Mo_1}$ and $\mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right)_{Mo_2}$ to be even closer together. Thus, if an optimal model Mo_1 with a predetermined $k_1 = k$ is known, the requirement

$$\mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right)_{Mo_1} \approx \mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right)_{Mo_2} \tag{12}$$

almost reduces the 2-parameter search for an optimal model Mo_2 with constraint $k_1 = k + \Delta$ to a 1-parameter search. It is only "almost" because '$\approx$' is not '$=$'. '$=$' is not even wanted, the formula just leads to promising starting points for iteration.

The following table shows parameters and the expected value $\mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right)$ of the best determined model for BSM parameters k_1 from 25 to 110 (DAX call options 05.07.2016).

BSM	k_{2d}	$q_{1,2}$	$\mathbf{E}_Q\left(\log\left(\widetilde{S}_1/\widetilde{S}_0\right)\right)$
25	1043	0.08663	-1.31795E-05
30	613	0.04123	-1.07231E-05
35	519	0.02563	-1.01865E-05
40	479	0.01788	-9.96467E-06
45	456	0.01335	-9.83428E-06
50	441	0.01043	-9.75004E-06
55	431	0.00841	-9.69863E-06
60	424	0.00694	-9.66491E-06
65	416	0.00587	-9.61842E-06
70	411	0.00502	-9.59267E-06
75	407	0.00436	-9.58296E-06
80	402	0.00383	-9.55494E-06
85	399	0.00339	-9.54757E-06
90	394	0.00304	-9.52382E-06
95	390	0.00274	-9.5065E-06
100	387	0.00248	-9.49566E-06
105	382	0.00227	-9.47282E-06
110	377	0.00209	-9.45242E-06

(13)

Remark 15 *a) For models with varying time steps this works too, as it is always assumed that the model is a good approximation of its refinement with constant time steps.*
b) General augmented models are asymptotically steady-stated. The quality of equation (12) then depends on the speed of convergence.

6.4 Best Initial Distribution of Regimes

Given all other parameters, it is sufficient to find the minimum of a quadratic function to determine the initial distribution (q_1, q_2) of regimes, which minimizes the norm distance.

Let CC_e be a European option with maturity T and let $\widetilde{CC}_e\,(t\,|\,q)$ be the value of this option at time t $(0 \leq t \leq T)$ in the model with $q_1 = q$. By formula (6)

$$\widetilde{CC}_e\,(t\,|\,q) = q\;\widetilde{CC}_e\,(t\,|\,0) + (1-q)\;\;\widetilde{CC}_e\,(t\,|\,1)$$

where $\widetilde{CC}_e\,(t\,|\,i)$ $(i=0,1)$ is the value in the pure model that starts in Z_{2-i}. Let n_f be the number of maturities in the sample and n_j be the number of options with maturity "j" in the sample. Then

$$\beta_j = \frac{1}{n_f}\frac{1}{n_j}$$

is the weight of the options with this maturity in the formula of the square of the norm distance:

$$\begin{aligned} ND^2 &= \sum_{j=1}^{n_f}\sum_{i_j=1}^{n_j}\beta_j\left(CC_{i_j}(q) - CC_{i_j\,M}\right)^2 \\ &= \sum_{j=1}^{n_f}\sum_{i_j=1}^{n_j}\beta_j\left(\left[q\,CC_{i_j}(0) + (1-q\,)\,CC_{i_j}(1)\right] - CC_{i_j\,M}\right)^2 \end{aligned}$$

$(CC_{i_j}(q) = \widetilde{CC}_{i_j\,e}\,(0\,|\,q\,)$ and $CC_{i_j\,M}$ the market price of option CC_{i_j} at $t=0)$. The right hand side is quadratic in q:

$$ND^2 = c_2\,q^2 - 2c_1\,q + c_0$$

where

$$\begin{aligned} c_2 &= \sum_{j=1}^{n_f}\sum_{i_j=1}^{n_j}\beta_j\left(CC_{i_j}(0) - CC_{i_j}(1)\right)^2 \\ c_1 &= \sum_{j=1}^{n_f}\sum_{i_j=1}^{n_j}\beta_j\left(CC_{i_j}(0) - CC_{i_j}(1)\right)\left(CC_{i_j\,M} - CC_{i_j}(1)\right) \\ c_0 &= \sum_{j=1}^{n_f}\sum_{i_j=1}^{n_j}\beta_j\left(CC_{i_j\,M} - CC_{i_j}(1)\right)^2 \end{aligned}$$

So, the minimum of a quadratic function on the interval $[0,1]$ has to be determined, which either is one of the boundary points or the only critical point $q_0 = c_1/c_2$. As $Z_1 \neq Z_2$ in all sensible cases, c_2 is always positive. From this it follows that $q_0 \in (0,1)$ if and only if $c_2 > c_1 > 0$. This need not be the case, but if it is because of convexity q_0 always minimizes ND^2 and hence ND.

7 DAX Options and the Brexit Referendum

7.1 Introduction

On Thursday, 23rd of June 2016, the so-called EU referendum or Brexit vote took place in Great Britain. The eligible voters of the United Kingdom were asked to decide whether Great Britain should leave the European Union or remain a member. The referendum was originally initiated by prime minister David Cameron of the Conservative Party to get strong support for a basically proEuropean political attitude. While in the first months of 2016 it seemed to add up to this, the opinion polls became closer and closer as the election day came nearer [9], and finally the result was that 51.89% voted for Britain to leave the EU.

This was a political decision, of course, but surely one with a huge impact on financial markets - at least in Europe. For example, what London's role as a financial market will be, is a question that instantly arises.

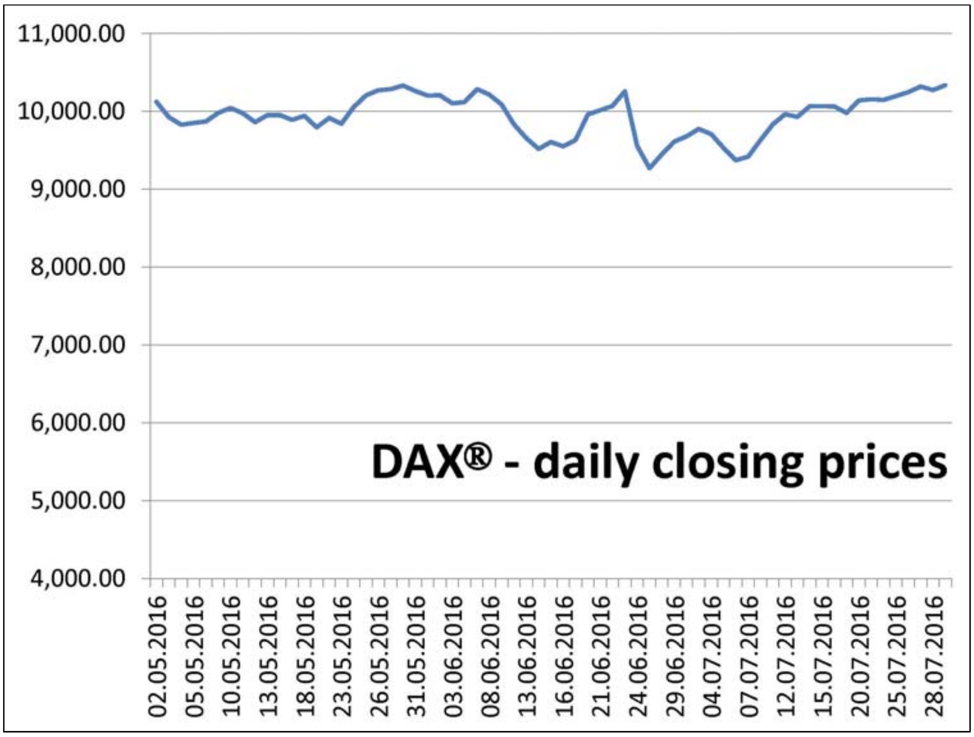
The growing uncertainty before the voting day in 2016, the result of the vote and the new situation thereafter must have left traces in many financial data - not only in Great Britain, but also in other European countries, especially in France and Germany.

In this chapter we present the results of a study of Eurex DAX® call option prices in a period from the 02nd of May 2016 to the 29th of July 2016. This period contains 65 days with published daily market statistics by Eurex, among them 1 official bank holiday in Germany (Whit Monday; stock exchange closed) and 2 partial bank holidays (Ascension Day and Corpus Christi; reduced stock exchange). All these days have been included.

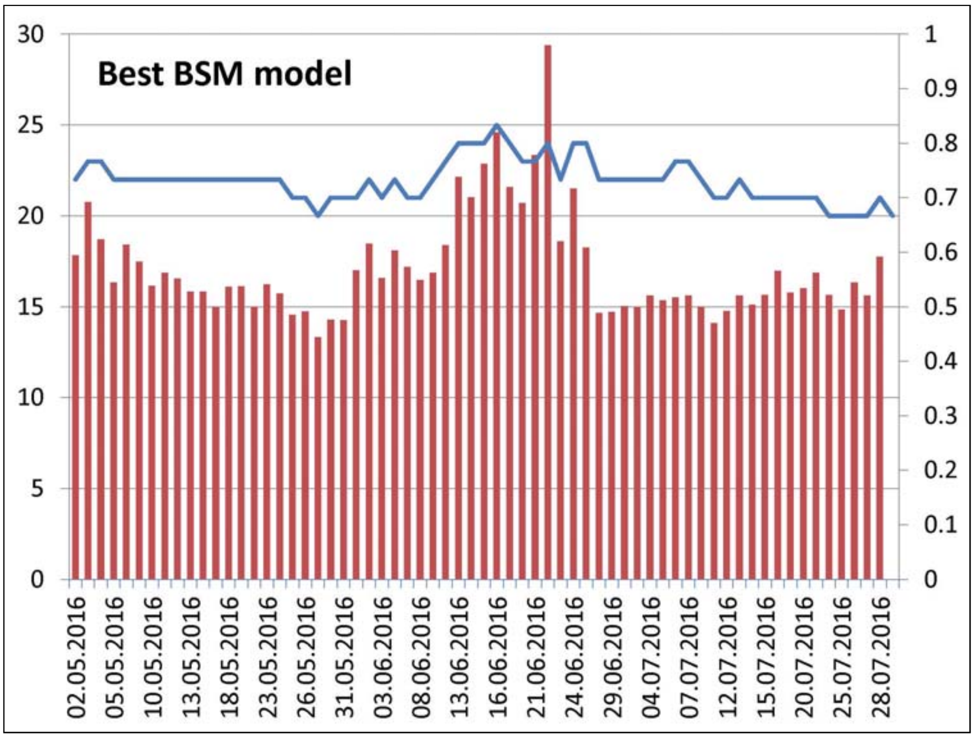
To discover the influence of the Brexit vote on call option prices is only the second of two purposes of this study. The first one is to basically find out, to what extent the selected skew tree model is able to produce call option prices of a real market - in calm and in rough times. This question again has two components: The first one is how precise (measured in norm distance) real implied volatility surfaces can be generated by the model, and the second one is how stable the parameters are.

We begin with some fundamental data. The following figure shows a chart of the daily DAX closing prices from May to July 2016. On the first day of the period (02.05.), the DAX started with a closing price of 10,123.27. Some lesser ups and downs occured until the end of May, but by the 13th of June at the very latest a more volatile phase was entered. On the 23rd of June, the voting day, the closing price was 10,257.03 and hence even a little bit higher than on the first day. But then the index lost nearly 10% within the next two trading days (6.82% on 24.06.) and closed at 9,268.66 on the 27th of June. But these losses did not last long. At the end of July, the DAX closed at 10,337.50, the highest closing price of the whole period.

[9] see https://whatukthinks.org/eu/



DAX - daily closing prices from 02.05.2016 to 29.07.2016



Volatility in % (line and left scaling) and norm distance in % of DAX

A first orientation about option prices can be obtained by a look at the best BSM models (figure on previous page bottom). They were determined on the basis of the same market prices as the main study was, which means that on each trading day at least 372 option prices were included (15.07.) and at most 520 (15.06.). The average is 444 prices. More details can be found in the appendix A.1.

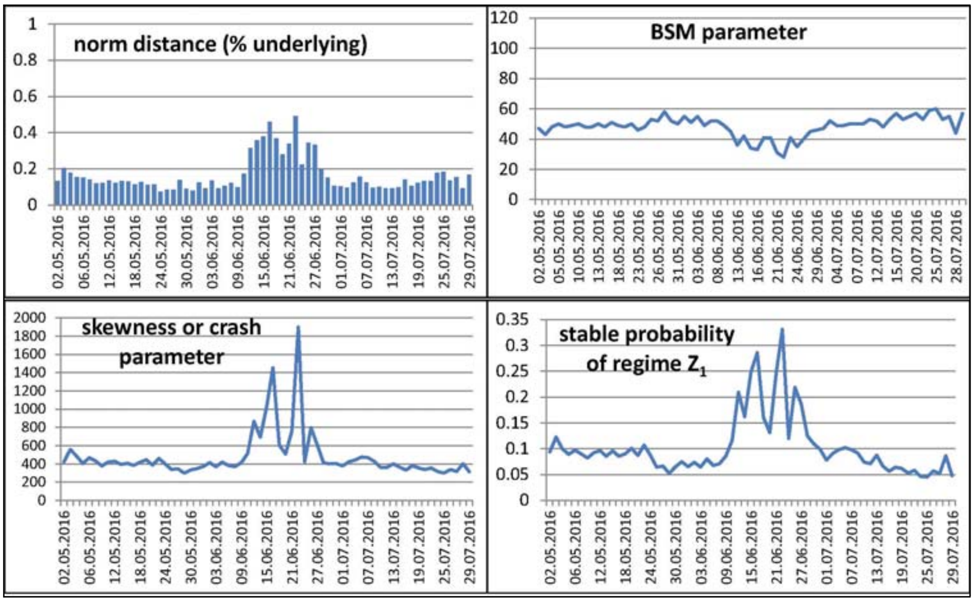
Description of the result: the volatility of the best BSM model was 22-23% from the 02nd of May to the 24th of May. Then it even sank to 20-22% from the 25th of May to the 09th of June. From the 10th of June to the 27th of June the volatility was never less than 23% and never greater than 25% - with the exception of the voting day itself, when the value was 22%. From the 28th of June on, the best BSM model volatility was mostly lower than 23 (only 2 exceptions) and afterwards sank further to 20-21% in the last 12 trading days. These data suggest that by the end of July to a great extent normality had returned.

The columns of the diagram show the norm distance between the best BSM model's prices and the market prices. In advance, this distance was not suppposed to be very small, and indeed, it isn't. Hardly ever was the norm distance less than 0.5 percent of the underlying's price. On many days - especially near the voting day - it was much higher and worst of all the value was almost 1 percent of the DAX closing price on the 22nd of June. These values have to be undercut very clearly by a model that claims to be able to reproduce market prices.

On the other side, a norm distance of 0.1 percent or better is in the range of the local volatility model [Dupire 1994], a model that is proven to be able to reproduce every continuous arbitrage free IVS (at the cost of some desirable regularity and dynamical properties of the process (cf [Musiela Rutkowski 2007] p 256 and [Hagan et al 2002]).

7.2 Steady-State Models

As written above, in this section the search for the best fitting model of a trading day is restricted to models that start with the equilibrium distribution of the regimes Z_1, Z_2, and have $q_{1,2} = 0.1$ ($q_{1,2} =$ the probality that Z_1 (as actual regime) is left in the model's smallest time step).
And this is what has been found:

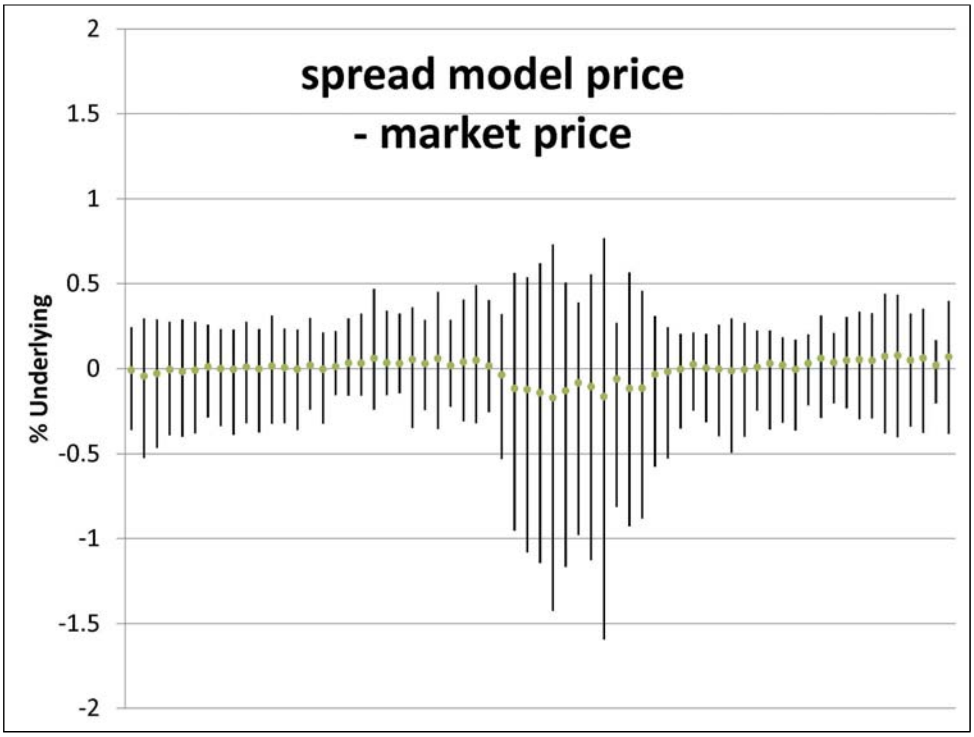


Best fitting steady-state models

The diagram at the top left shows the norm distance between the option prices of the market and the best model found for each trading day. The other 3 diagrams show the parameters of these models: BSM parameter k_1, crash parameter k_{2d} and - instead of $q_{2,1}$ - the more vivid stable probability π_1 of Z_1. This probability is also the portion of time that the (risk neutral) system will approximately be in regime Z_1 in the long run.

The figure on the next page provides another view of the model-market fit. It shows minimum, maximum and average of the difference model price minus market price. For each trading day all of the (at least 372) price differences lie in the plotted range. This is important as not all market prices refer to real trades (cf. A.1).

Both approaches lead to the same conclusion: the total period is divided into three subperiods. In period 1 and 3 for each trading day parameter sets have been found that define good to very good replications of the market prices of that day. In the second period, which lasts from the 13th to (about) the 28th of June, this is not the case. From 4 of the 5 diagrams it can be seen at first glance, that something unusual was happening here. We shall return to this later, but first discuss period 1 (02.05. to 10.06.) and 3 (29.06. to 29.07.).

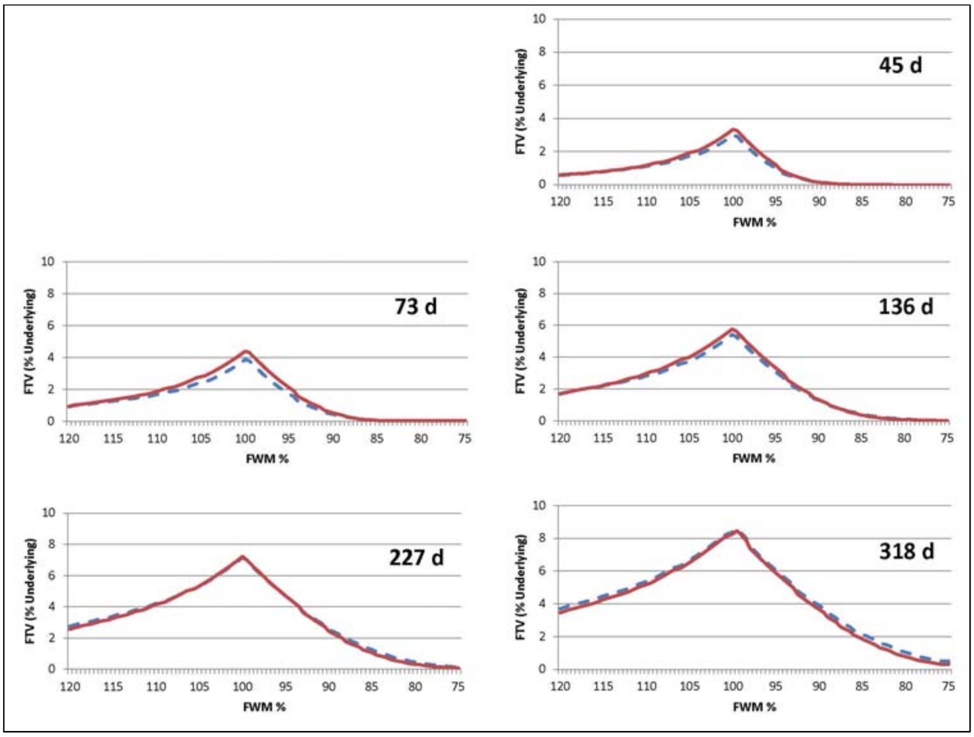


Max, min and average price difference

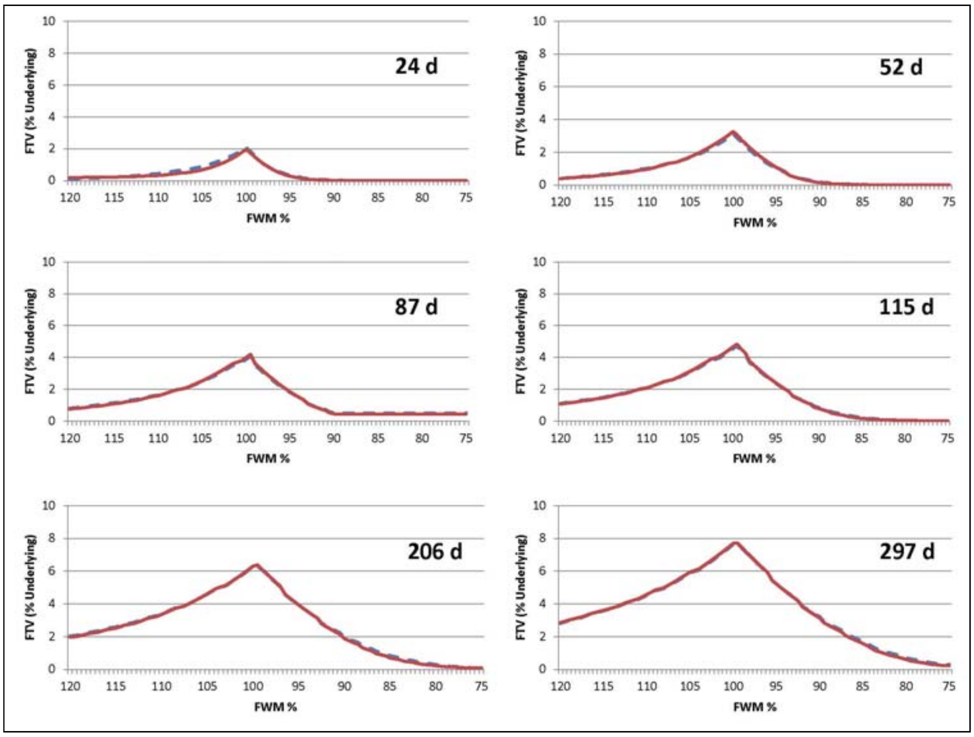
In these periods norm distances and spreads are small and the parameters vary only moderately. This can be read from the diagrams and is expressed in the numbers of the following table:

	per 1				*per 3*		
	min	*Ø*	*max*		*min*	*Ø*	*max*
norm dist. (% u.)	0.0713	0.1202	0.2015		0.0890	0.1231	0.1812
BSM parameter	43	49.8	58		44	52.3	60
crash parameter	299	407	558		300	377	475
π_1	0.0522	0.0850	0.1226		0.0452	0.0737	0.1096

An impression of the congruence between market and best model perhaps can be given by the next two figures, which show the prices of the worst (03.05.; $ND = 0.2015$)) and the best (24.05.; $ND = 0.0713$) fit in the sense of norm distance (ND). The situation on the 05th of July, as presented frequently in the preceding chapters, represents a fit of midrange quality among the trading days of periods 1 and 3.



Worst best fit (03.05.2016)

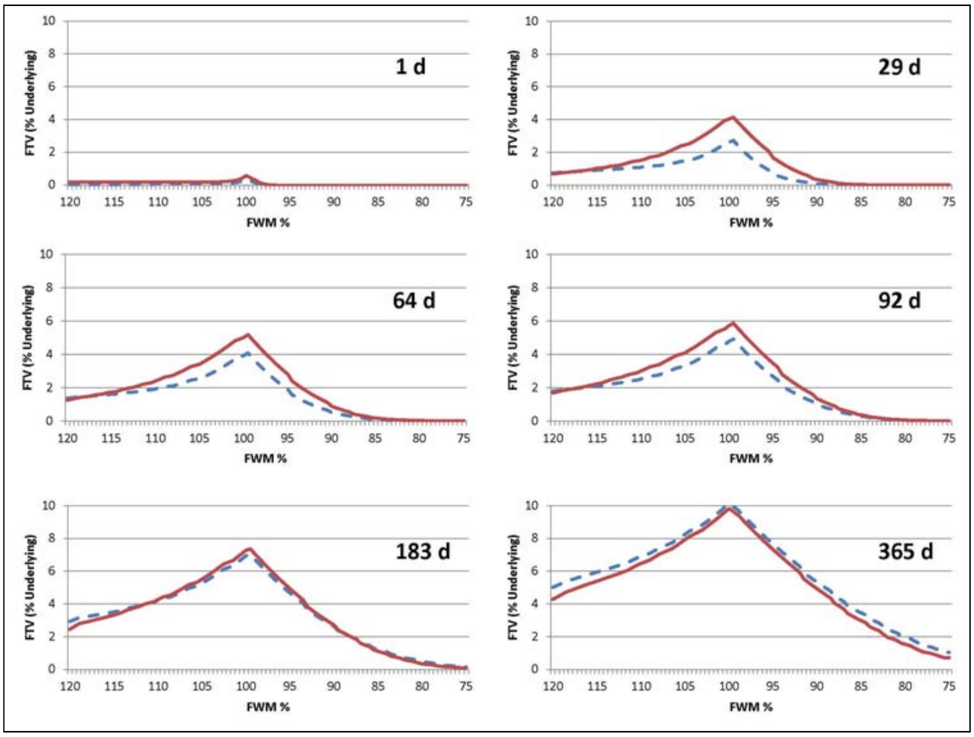


Best best fit (24.05.2016)

Now, after having registered the almost perfect fit of the 24th of May we turn our attention to period 2, which consists of the two and a half weeks from the 13th of June to the 28th of June - almost centered by the voting day (23.06.). The picture is quite different. Instead of a convincing model-market-fit a picture of disharmony appears. The skewness parameter explodes and crashes down again several times, as does the stable Z_1-probability. But all these efforts are in vain and cannot help the norm distance to raise up to 0.4878 on the 22nd of June, the eve of the vote. The average value during the period is 0.3353. On the voting day itself it is like the quietness in the center of a hurricane: the parameters are "normal" and the norm distance model / market is 0.2223 - a value that is nearly in the range of periods 1 and 3. One day later, the norm distance between best fitting model and market is up again at 0.3430.

As already written, the worst model / market fit is that of the 22nd of June. The second worst is on Thursday, the 16th of June, with $ND = 0.4583$. This one has been picked to show in detail, as this trading day has all the typical properties of the prices of this period (figure on this page; maturity March 2017 not shown, but included in the investigations). Moreover, this trading day can be used to show an additional effect.

At first, the typical behaviour: A comparison of market prices and prices



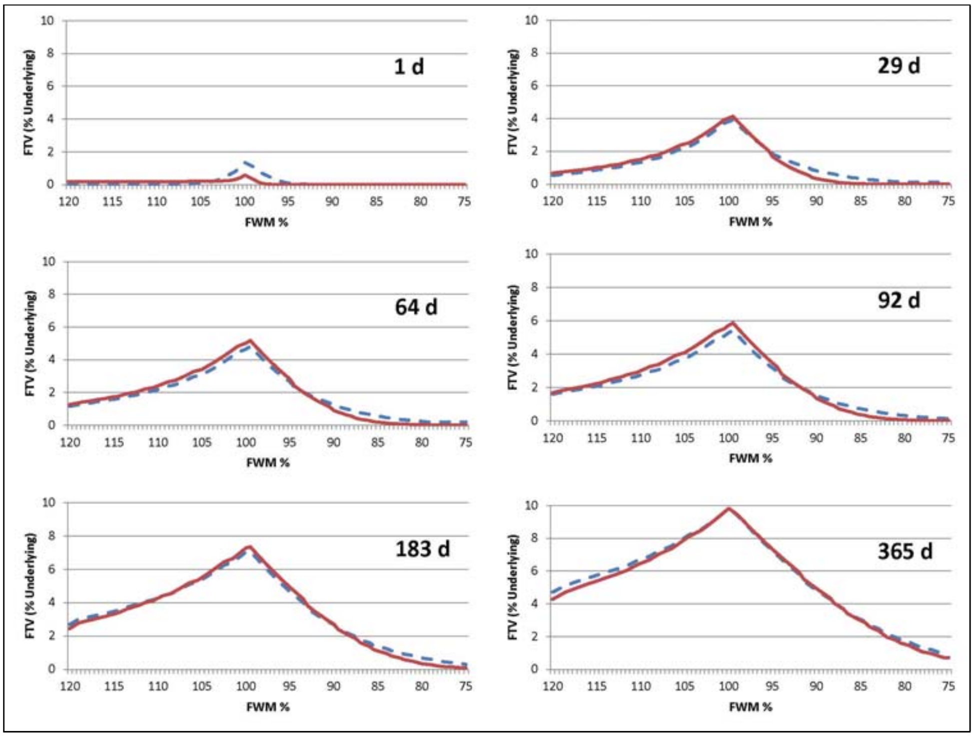
2nd worst best fit in period 2 (16.06.2016) of the best fitting steady-state model shows that in the market short running options (especially maturity in Jul (29 d) or Aug (64 d)) are very expensive, much more than long running ones. The market expected the Brexit vote to have much more impact on financial markets in the near than in the remote future. The price level as a whole is heightened (as can already be seen from the best BSM model, see page 57). This higher price level is not a problem for a steady-state model, but the velocity in which options get cheaper as maturity increases is. This apparently cannot be produced by one of the steady state models - presumably because of their stationary increments. Every suitable model either has too cheap short running options, or too expensive long running ones - or both. The best fitting model can only be a compromise and has no chance of being a true good fit. The question arises if this is a defect of this class of models or if the market is to blame, i.e. comprises temporary arbitrage possibilities in this special situation. In the next section general augmented models are studied, which can answer this question to a great deal.

The announced additonal effect: In the figure of the 16th of June the shortest running option (Jun) has just one day left to maturity and hence was known to be expired long before voting day. So there is no reason visible why this option should be influenced by the Brexit vote. And so it is. Only the July to September options are much more expensive in the market than in the model.

7.3 General Augmented Models

The very exceptional and known in advance event of the Brexit vote surely suggests the assumption that in the days surrounding the event some special effects were active on option prices. It was tried to get better results for phase 2 (13.06.2016 - 28.06.2016) by allowing general augmented models (with $q_{1,2}$ not necessarily equal to 0.1), thereby following the idea that a special situation is given near the trading day, but eventually returns to normal times afterwards. General augmented models have asymptotically stationary increments.

And indeed, this approach worked. In case of the 16th of June (cf last section) the norm distance could be improved from 0.4583 to 0.2387. The figure on page 64 shows prices resp. forward time values of the model.



16.06.2016: FTV of best general augmented model (dashed lines)

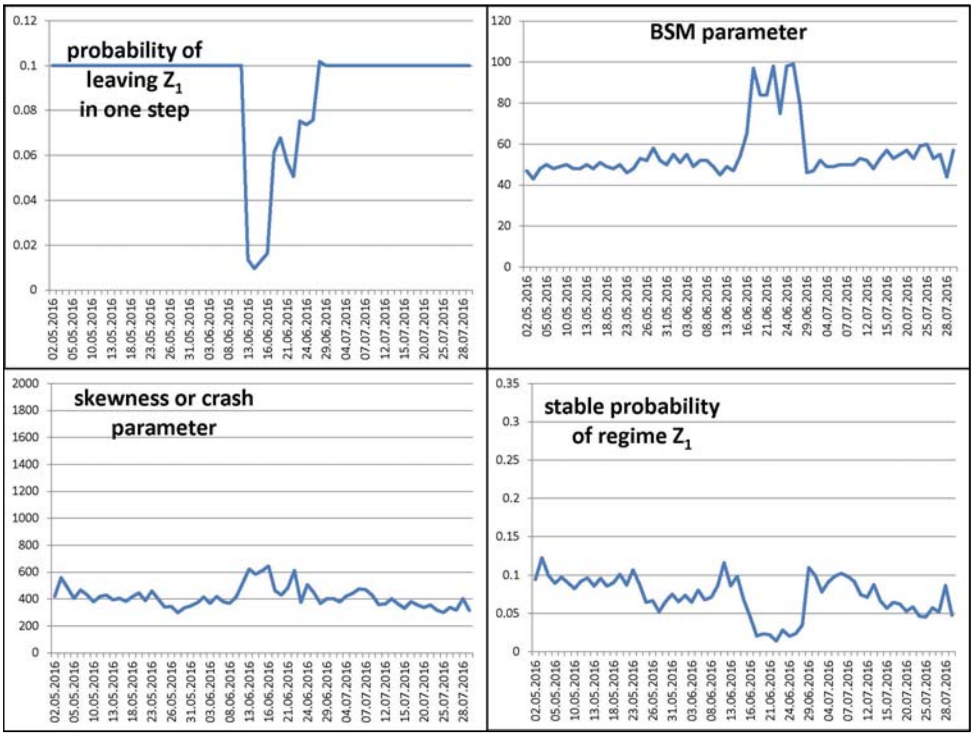
The parameters of this model in comparison to those of the best steady-state model are as follows:

	k_1	k_{2d}	$q_{1,2}$	$q_{2,1}$	q_1
steady-state	33	1452	0.1	0.04001	π_1
gen. augm.	65	645	0.0164	0.00078	1

($\pi_1 \approx 0.2858$) Apparently $q_1 = 1$ in combination with $q_{1,2} = 0.0164$ makes Z_1 rule the beginning phase of the process. The expected value of the first time, Z_2 is entered, is $\Delta t_0 \,/\, q_{1,2} \approx 6.8$ days. Later on, the process stays in regime Z_2 most of the time (in the general augmented model Z_1 has a stable probability of no more than 0.0454). So the BSM parameter $k_1 = 65$ makes the prices of options with early maturity rise (as desired), but plays only a supplementary role in the long run. In this way, the prices of model and market are brought closer together. But there are some unwanted spin-offs. The assured start in Z_1 in combination with an unlikely leaving of this regime makes it almost impossible to impose skewness to the process in the beginning. In this case it is very similar to a BSM process, as can especially be seen with maturity

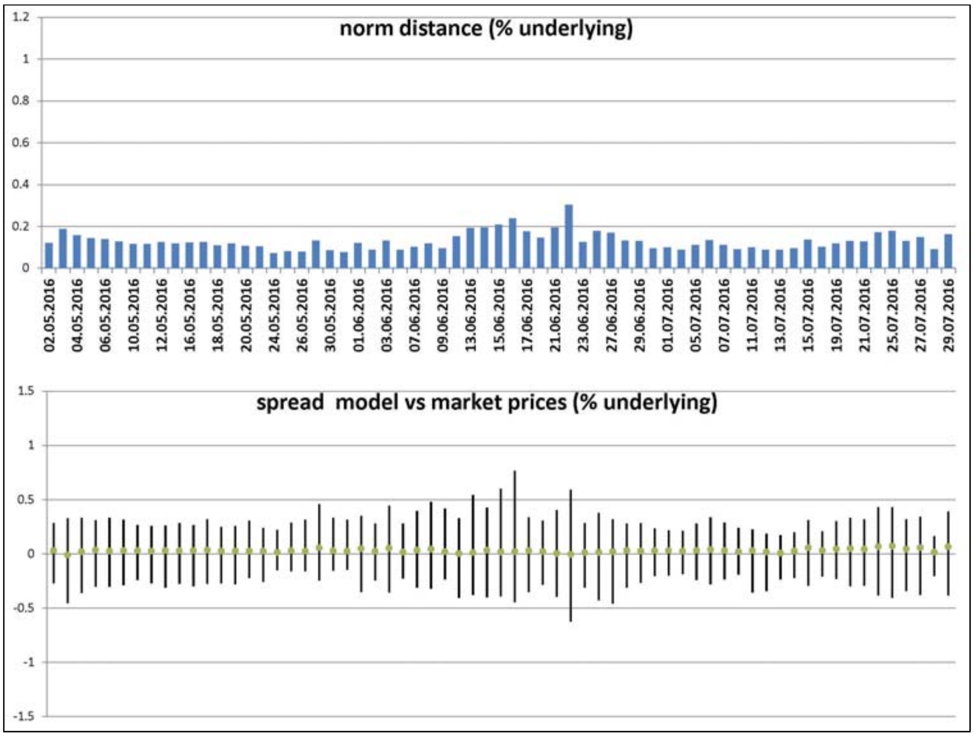
29 d. Although the prices of the general augmented model are pretty close to the market prices, the character of skewness is better represented by the steady-state model.

This is even more true for maturity 1 d, but cannot be seen very well because of the smallness of the time values and some side-effects of that. But the 1 d maturity option prices make another effect blatant: the model prices of these options are too high. So this is the way the phenomenon described at the end of the last section appears in this model, with just 2 regimes still fairly simple. The high volatility BSM model that is meant to generate high option prices for options that expire a short time after the voting day cannot also generate low prices for options expiring before. So, in this case, the little important accordance of the prices of 1 d options had to be sacrificed.



Parameters and related quantities of the best found models

The trading day 16th of June is typical for the whole period from 13th of June to 28th of June. With the help of general augmented models the average norm distance can be improved from 0.3386 to 0.1885. Only 3 days have $ND > 0.2$: 15th of June (0.2091), 16th of June (0.2387) and 22nd of June (0.3026). All models of period 2 have $q_1 = 1$, i.e. they start with a BSM regime (with high volatility from the 16th of June on - not on the first days of period 2!) that has a low stable probability. The probability of leaving this regime is low in the models near the beginning of period 2, but in the end of period 2 rises to the standard value 0.1 of the steady-state models. The skewness parameter k_{2d} is slightly increased but stays in "normal" regions and avoids the extreme heights of some of the steady-state models of period 2. The figure above shows the parameters (π_1 instead of $q_{2,1}$) of the general augmented models of period 2. For the trading days of period 1 and 3 the diagrams of this and the next page show the data of the best steady-state models - similar to page 59. For periods 1 and 3 no best general augmented models were searched for systematically. However, some attempts were made to find notably better general augmented models, but they were not successful (see next page for one exception).



Market prices vs prices of best models

The figure above consists of two diagrams. The upper one shows the norm distance between the market prices and the so far best found model prices. In the lower one the minimal, maximal and average spread between these prices can be found. For period 2 the values refer to the models detected by search in the bigger set of the general augmented models, for period 1 and 3 improved steady-state models are shown. 'Improved' means that almost all parameters are unchanged, only the initial distribution (q_1, q_2) is not the equilibrium distribution. Instead, it is the distribution that minimizes the norm distance between model and market prices under the constraint that the other parameters are fixed. This is meaningful, as it improves the average norm distance (period 1: 0.1154 instead of 0.1202; period 3: 0.1187 instead of 0.1231), can be done with little effort and does not modify the model in depth (cf Section 6.4). As usual, all values are to be understood as *% underlying*. Hence 0.2 means 20 Cents, if the underlying's price is 100 €.

7.4 Stability of Prices

So far it could be demonstrated that the skew tree model can reproduce the market prices of call options during the whole period from May to July for each trading day at least in a satisfactory, but mostly a good to very good way. This leads to the following benefit: First it shows that the market prices are close to an arbitrage free system because all the models are free of arbitrage possibilities. This does not exclude minor theoretical arbitrage possibilities in reality (as the difference between model prices and market prices is not zero), but it should not be easy to generate profit by them.

As a next step, a well fitting model can be used to determine prices of nonstandard options. At least these prices are arbitrage free (in combination with the underlying and all European call and put options), which is not a bad argument in favour of them.

Remark 16 *In this context one automatically thinks of path dependent options like barrier options, but they require some caution. If varying time steps are used in the implemented model - which will almost always be the case, as this is one of the key features that enable models like this to be implemented - one must bear in mind, that these models have very special paths. Only for a sequence of small time steps does the* $\widetilde{S}$*-path look similar to a path of a Brownian motion, for big time steps it looks more like a polygonal line. Brownian bridges should be helpful.*

Most of all it is desirable that a model tells how options can be hedged. We do not treat hedging in depth in this article, but some fundamental aspects of hedging in theory can be found in 5.4. Whatever a hedging strategy in a tree model may be, in its pure form it will consist of portfolios in the chosen hedging instruments that have to be adjusted in the trading times t_i. Surely, riskless borrowing/lending and the underlying asset can or even has to be among the hedging instruments. But as the models are incomplete, some additional instruments are needed - derivatives of the underlying which are traded on liquid markets are natural candidates.

In theory, every option pricing model has the requirement to be appropriate until the option expires. But as already shown by [Hull Suo 2002], hedging on this premise does not produce good results at all. In reality, it is a widespread practice to adjust a hedging portfolio by use of a current model (i.e. set of parameters) and then later on readjust it on basis of the model (set of parameters) that is valid then. This is done in the hope that this systematic friction does only produce white noise error and will eventually sum up to zero. Ideally, in the moment of system change, both the old and the new model have the same prices for all hedging instruments. This situation, of course, would be given if both models were the same, i.e. had the same parameters. The figures of this chapter show that the parameters sometimes are quite stable (period 1 and 3), but vary strongly in other times. Still, different parameters do not necessarily mean very different option prices or values of hedging portfolios. In theorem 12 of section 5.4 it is shown that a change of augmentation (reaugmentation) may change option prices, but does not change the value of a correct hedging portfolio.

It has been investigated how stable the model parameters and model option prices were locally in May, June and July 2016. From the second trading day on, the prices of the options in the foregoing day's best model have been calculated and the norm differences to the market prices have been computed. Observe that 'foregoing trading day' does not mean 't_{i-1} in model'.

The next table shows the average norm distance in the 3 periods as well as the overall average norm distance. The first line repeats the known results of the actual best models. The second line contains the norm distance between the market prices and the prices of the previous trading day's model. In this line, a trading day has only been included if that day and the previous trading day both lie in the considered period.

ND	per. 1	per. 2	per. 3	per. 1 - 3
actual model	0.1154	0.1885	0.1187	0.1300
previous day's model	0.1696	0.3780	0.1684	0.2090
prev. day plus reaugm.	0.1568	0.3222	0.1590	0.1892

The third line contains the norm difference between market prices and the prices of the previous trading day's model with reaugmentation, such that the norm distance to market prices is minimized. This is allowed and meaningful, as already pointed out in Chapter 5.4. In period 2 reaugmentation betters the average norm distance from 0.3780 to 0.3222. The biggest effect of reaugmentation has been detected on the 23rd of June (the voting day) when applied to the model of the previous day: more than 0.20. In period 1 and 3 the average norm distance has been reduced at least by more than 0.01% of the underlying or more than 5% of its own value. For the whole period 1-3 including all Brexit days by reaugmentation the average norm distance could be reduced to a value below 0.19.

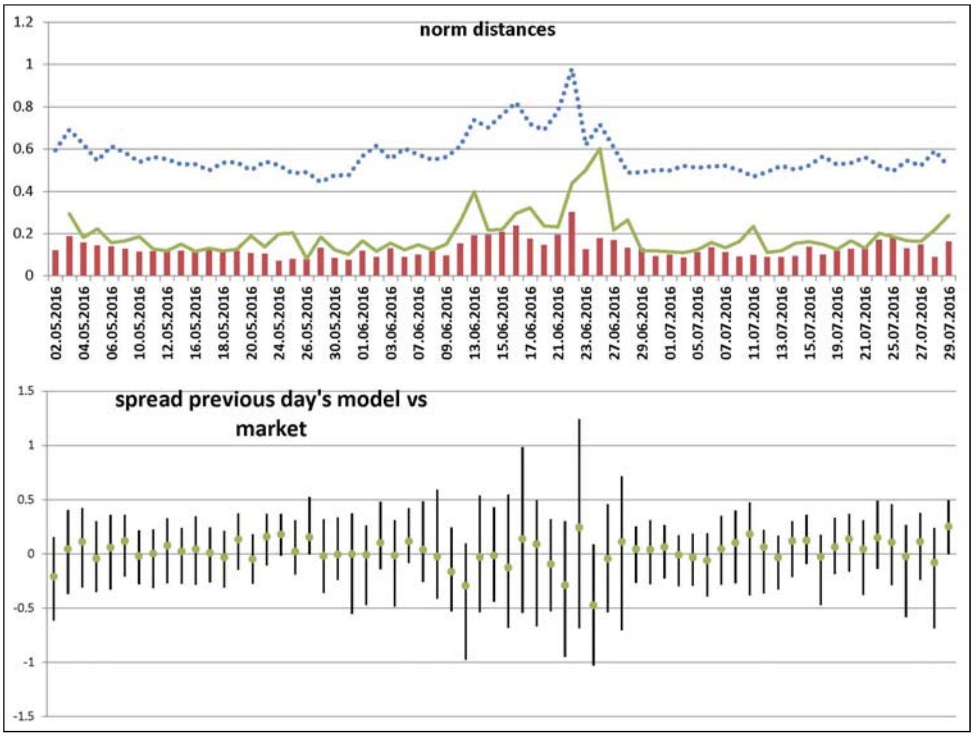
Comparing lines 1 and 3 shows that the reaugmented previous day's models lead to average norm distances that are at least 30% larger than the results of the actual models (in period 2 more than 70%). This is clearly worse than the actual day's models results, but it should be noticed that at least the results of per. 1, per. 3 and per. 1 - 3 are still good results.

No single model, i.e. single parameter set, has been found that, when applied to all days of period 1, leads to an average norm distance of 0.16 or better in period 1 - even if reaugmentation is permitted. In period 3 no single model with average norm distance better than 0.20 was found.

Although this search for a unique model for a period was not done thoroughly and there may very well be better unique parameter sets, it looks more promising to continue to use the actual day's best model to build up a hedging portfolio and then live with the differences to next day's prices. Moreover, this approach is easyer to practice.

Remark 17 *'Previous day' is not totally correct. It is not completely identical with the prices the model of the previous day had foreseen under the assumption that today a certain node is appropriate. 'Previous day' means the prices that the model with the parameters of the previous day (regimes and transition matrix) creates, when applied today. If variable time steps are used, that will not be completely the same. But the difference should be small.*

Remark 18 *A comparison with other models would be interesting, of course. There are the "classical" models of Heston [Heston 1993], Merton [Merton 1976] and Bates [Bates 1996], for instance, as well as younger approaches ([Cavet et al 2015] or [Hauser Shahverdyan 2015] for instance, and many others). Of special interest would be a comparison with the local volatility model of Dupire [Dupire 1994]. In theory, this model is not to beat under the aspect of reproducing a given arbitage free (continuous) implied volatility surface. In former studies, local volatility models with norm distance to* *market prices less than 0.1% of the underlying's spot price could usually be found ([Graf 2008], [Wilbert 2008]). As seen before, the skew tree model of this chapter comes close to this value, at least in normal times, but stays a bit behind. So in order to be competitive the skew tree model should be better in other disciplines - first of all in price dynamics. What justifies the hope for this? It is the same reason as with the classical models cited above: the high degree of regularity of the process. All skew tree models have stationary increments or at least asymptotically stationary increments. The local volatility model, on the other side, allows for all types of irregularities. In fact, this is the reason why this model is so flexible, but does not have convincing dynamics. Anyway, a systematic comparative study of this aspect has yet to be conducted.*



Norm distances and spreads in context with "yesterday‘s model" (see text)

We end this chapter with a short discussion of the two diagrams of the figure on this page. The diagram in the upper half contains norm distances between model prices and market prices. The solid line shows them for the prices of the previous day‘s best models, whereas the columns refer to the actual model. Many values of "yesterday's model" are ok ($ND < 0.2$), but there are some bad exceptions, the worst being on the 24th of June ($ND = 0.6017$), followed by the 23rd ($ND = 0.5015$), the 22nd ($ND = 0.4368$) and the 13th of June ($ND = 0.3980$). It is no surprise that the voting day and it‘s direct neighbours are on this list. At least, “yesterday’s model” is always clearly better than the actual day‘s best BSM model (dotted line).

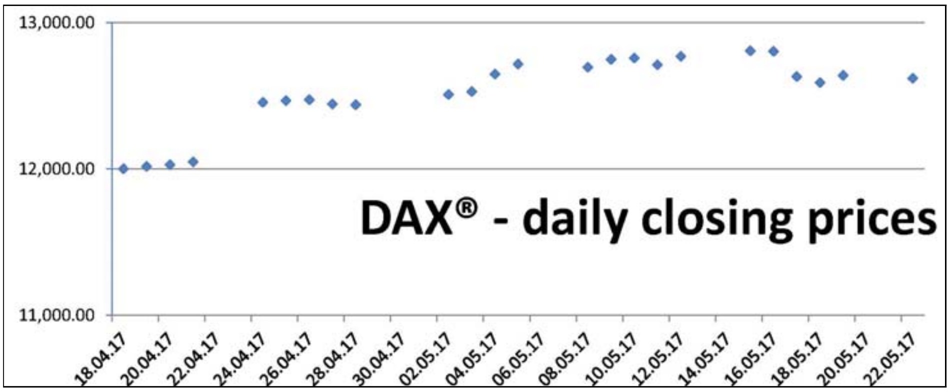
The lower half diagram shows the minimal, maximal and average difference between the prices generated by the previous day‘s model (reaugmented) and market prices. As usual, all values are ‘% underlying’. The extreme values of min and max are -1.0283 (24.06.) resp. 1.2412 (23.06.). All other values of min and max are between -1 and +1, the average value of the difference (max - min) of a trading day is 0.73 (0.62 with best models instead of “yesterday’s models”). The total average value of the price differences is 0.0210.

8 The French Presidential Election 2017

8.1 Political Proceedings and the DAX Charts

In 2017, ordinary presidential elections took place in France. The first ballot was on Sunday, 23rd of April with a run-off ballot between the two candidates with the highest number of votes to follow two weeks later (07th of May). Five years ago, in 2012, François Hollande of the socialist party (PS) had been victorious in a traditional duel against the candidate of the conservative *Union pour un mouvement populaire* (UMP), but this time the situation was different. In 2017, the traditional parties had lost much popularity and their canditates seemed to have no chance of winning the election. Instead, new “movements” were founded, among them *En marche* (social liberal, candidate Emmanuel Macron) and *La France insoumise* (far left, candidate Jean-Luc Mélenchon). Moreover, the opinion polls indicated that Marine Le Pen (*Front National*, far right) had good chances of winning the first ballot.

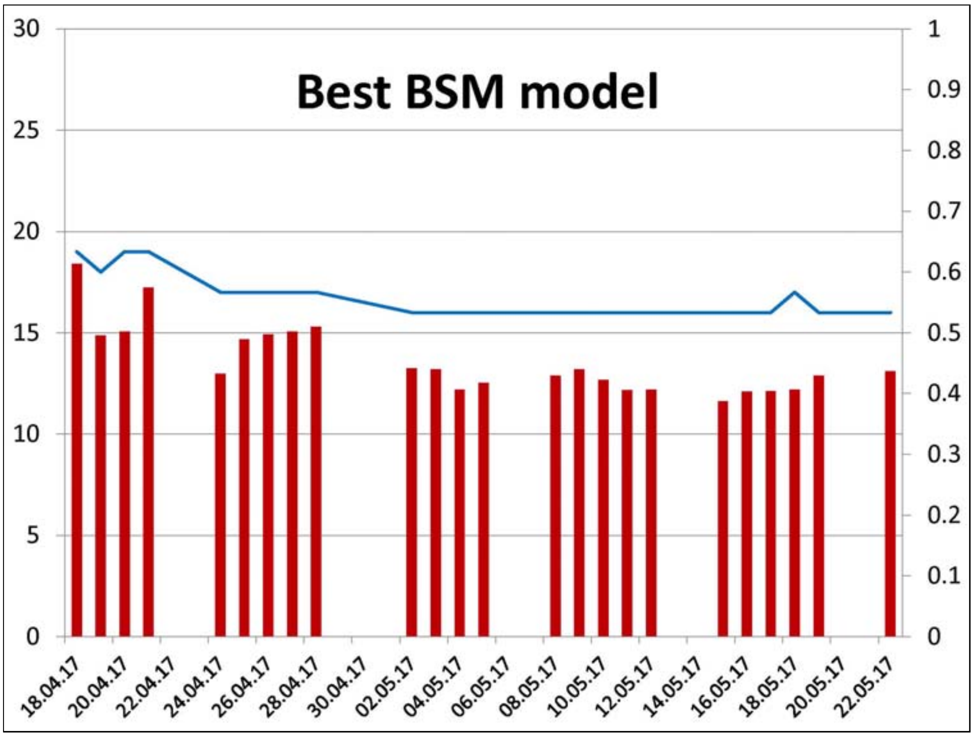
No doubt there was something special about the French presidential election 2017, but, at the same time, it was was much more of a “normal” event than the Brexit vote. But like the Brexit vote it was a political event that was likely to influence the financial markets not only in France but also - among other countries - in neighbouring Germany, a strong political partner. So the same investigation of DAX option prices was carried out as with the Brexit vote. Within the same model type of 2-regime skew tree models the best possible reproduction of Eurex DAX call option prices was searched for each trading day in the period from the 18th of April to the 22nd of May. Again there were the two aims to - first - find out how good and how stable the option prices can be reproduced by this model type and - second - what information can be obtained from these models about the behaviour of the actors in the financial market. A third aim was to identify differences between the periods 2016 and 2017 - if they existed.



DAX - daily closing prices during the time of the French presidential election 2017

The period contains 24 trading days, the only holiday was the 1st of May. A first impression of the period is given by the closing prices of the DAX (figure above) and the best BSM models (next page). The index DAX is about 2,000 points higher than in the Brexit days. There is a big rise after the first voting day: From 12,048.57 on 21st of April 406.41 points (3.37%) are gained to a closing price of 12.454,98 on Monday, the 24th of April. This was a reaction of the stock exchange markets to the results of the first voting day, which was won by Emmanuel Macron (24.0%) followed by Marine Le Pen (21,3%) and saw Jean-Luc Mélenchon (19,6%) failing to qualify for the second ballot. It had been feared before, that the extreme candidates Le Pen (right) and Mélenchon (left) could be the winners of the first voting. That presumably would have put France in a chaotic political status with unpredictable implications.

The second voting two weeks later did not cause any visible reaction on the DAX closing prices. The financial markets supposed Macron to cruise to victory, which he did (66.1% vs 33.9%). The friendly mood of the market lasted until the 16th of May. Then a slightly retreating market could be observed.

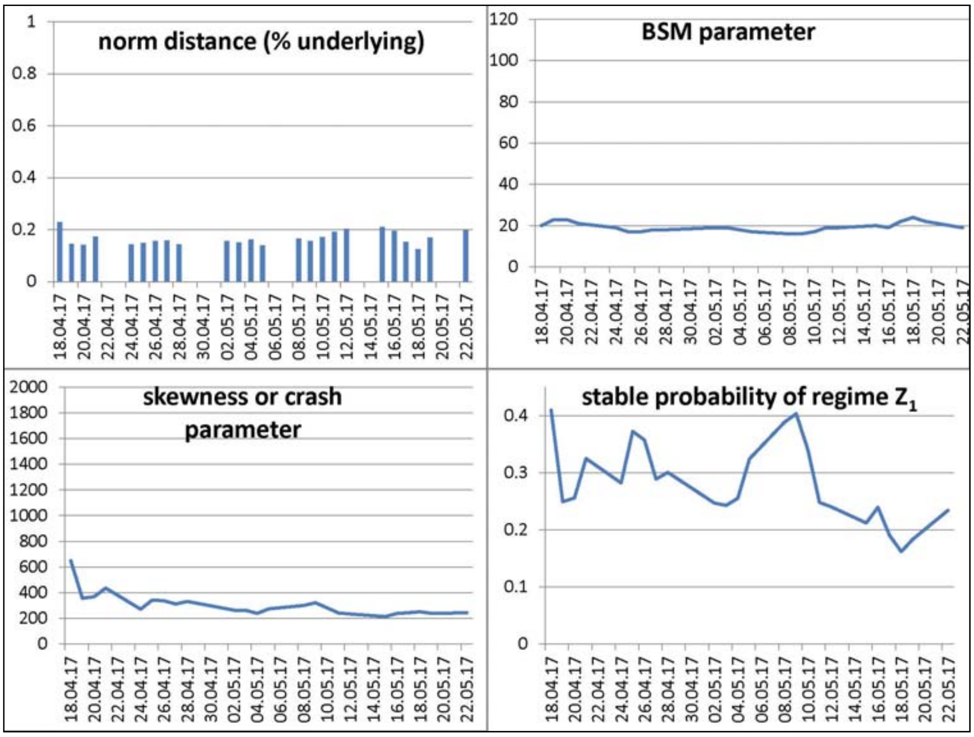


Best BSM model: volatility parameter and norm distance to market prices

Compared to the phase of the Brexit vote, the option courses were clearly less volatile. This is underlined by the best BSM models. The volatility parameter of these models is less than 20% throughout and very stable (solid line and left scaling). From the 24th of April onwards the only values are 16% and 17%. Moreover, the norm distance between the market prices and the prices of the best BSM models is only 0.436% of the underlying's price on average (0.570 in the Brexit period). This means that the implied volatility surfaces bear less smile/smirk than those of the Brexit days. But still, 0.436% is not a good value. It does not make the best BSM model a candidate for an overall option pricing model. It would be disappointing if the skew tree model could not produce much better values.

8.2 Best Skew Tree Models

Just as in the Brexit investigation, as a first step a best fitting steady-state model (with additionally $q_{1,2} = 0.1$) was searched for each trading day. After that it was tried to find better suited models by allowing general augmented models. Further details can be found in the appendix.



Norm distance to market prices and parameters of the best found steady-state models

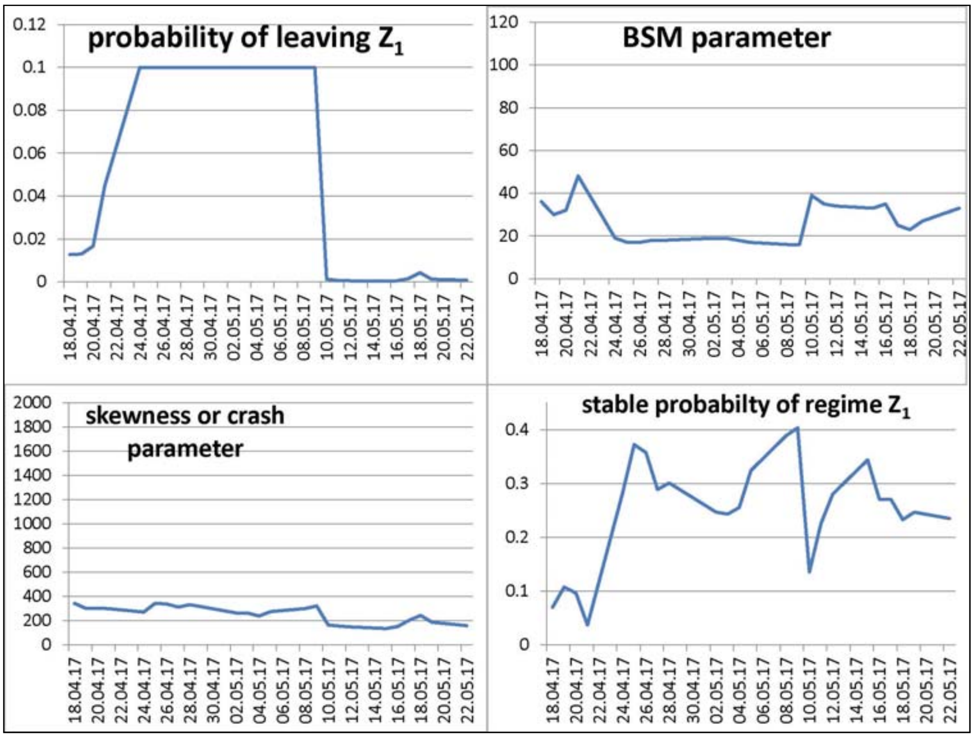
This figure shows the characteristic data - norm distance to market prices and parameters (or related values) - of the best fitting steady-state models. With the exception of the stable probability each diagram has the same scale as the corresponding one on page 59.

The situation is quite different to that of the Brexit study. The norm distance to market prices is fairly constant. It is never much bigger than 0.2 (% of the underlyings price) and almost always less than this value. But on the other side, no value is less than 0.1. The average value is 0.167.

Two parameters behave very regularly. The BSM parameter is always close to 20 and slightly larger than the volatility of the best BSM model. The crash parameter has its highest values before the first ballot and then slowly keeps sinking from about 350 to near 200.

Only the stable probability of regime Z_1 looks quite volatile, but tendentially is larger than in the Brexit case. Due to the comparatively low values of the BSM parameter, a longer stay in Z_1 does not automatically result in high option prices.

Since there are no really bad results and the norm distance to market prices is always at least twice as good as with the best BSM model, one might be happy with the stable state models. But it is possible to better the fit with general augmented models notably - not for each trading day, but especially on those days with comparatively large norm distance to market prices. As a result of optimization efforts the whole period has been divided into 3 subperiods and in two of these (period 1: 18.04. - 21.04; period 3: 10.05. - 22.05.) the steady-state models have been replaced by better suited general augmented models.

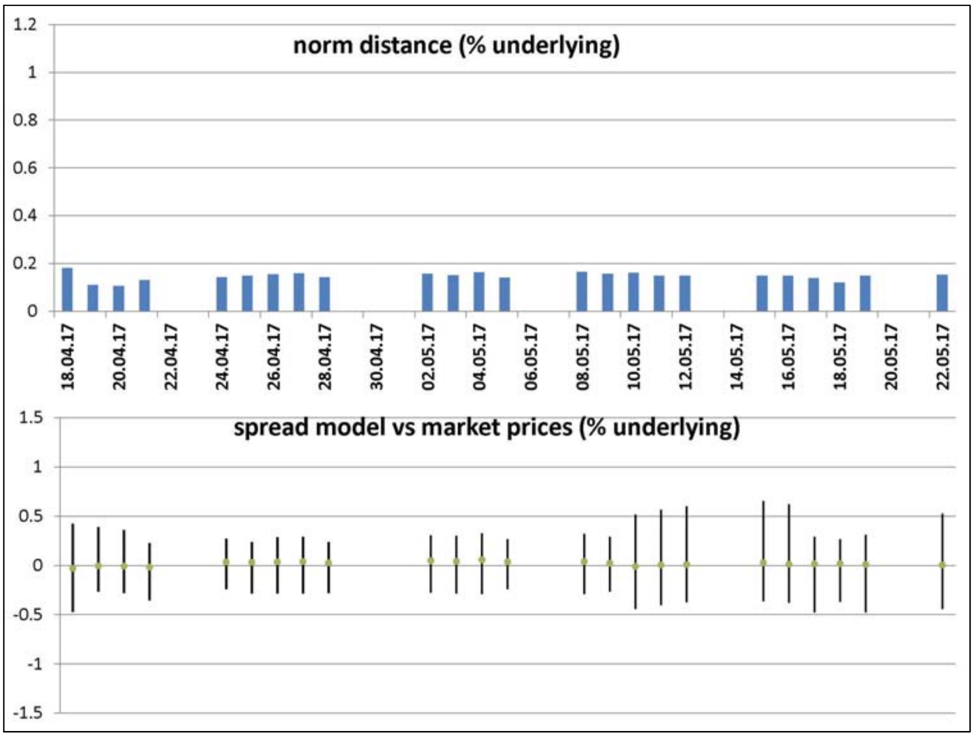


Parameters or related values of the best detected general models (French election)

For period 2 (24.04. - 09.05.) no significantly better general augmented models were found, which justified holding on to the best detected steady-state models - reaugmented (although this did not improve the result very much here). All in all by this operation a norm distance clearly better than 0.2 [10] could be realized for each trading day and the average norm distance could be improved to 0.147 - a value, that is pretty good, but not as good as

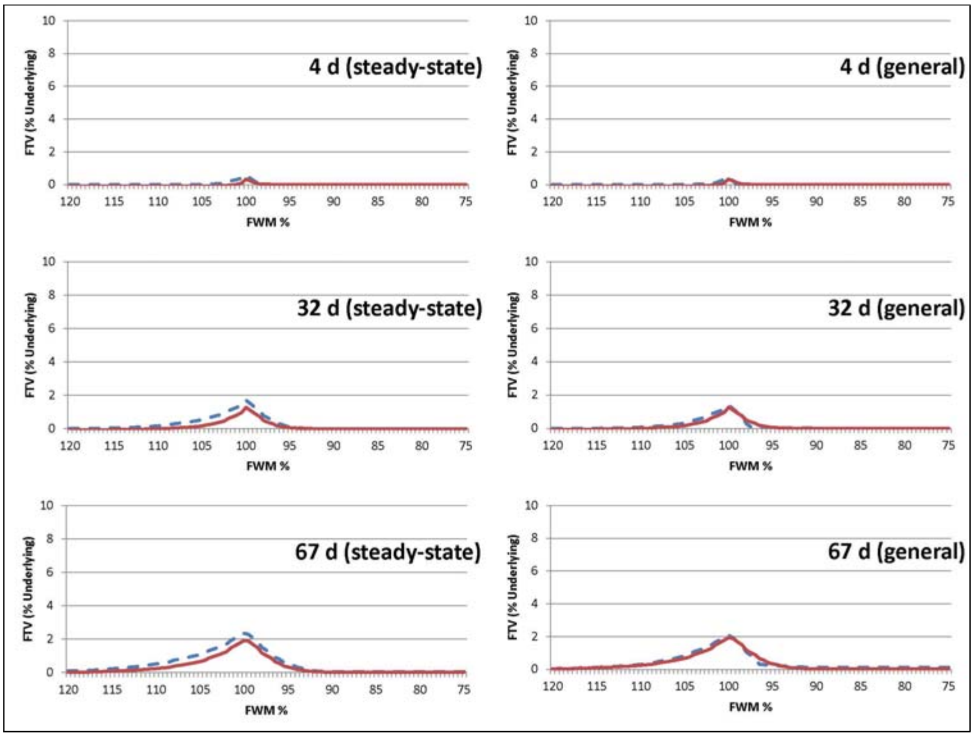
[10] As always, norm distance (ND) is expressed in percentage of the underlying's price

the final overall average value in the Brexit study (0.130, see page 68). The figures on page 75 and 76 show the same data for the French presidential election period as the figures on the pages 65 and 66 do for the Brexit vote study.



French election 2017: best found models

The reason why the approximation of the market prices could be improved by general augmented models is a different one for period 1 than for period 3. In period 1 - the time before the decisive first ballot - it is similar to the Brexit voting days: short running options are expensive, long running ones are not - and the steady-state models cannot handle this perfectly. In period 3 - after the second ballot - the situation is inverse: short-running options are comparatively too cheap compared to long running ones to reproduce this situation with the stationary-increment steady-state models (at least those with $q_{1,2} = 0.1$). General augmented models can overcome this problem by starting with regime Z_2, but there is a price to pay: a skewness kink in the model prices that the market prices do not show.

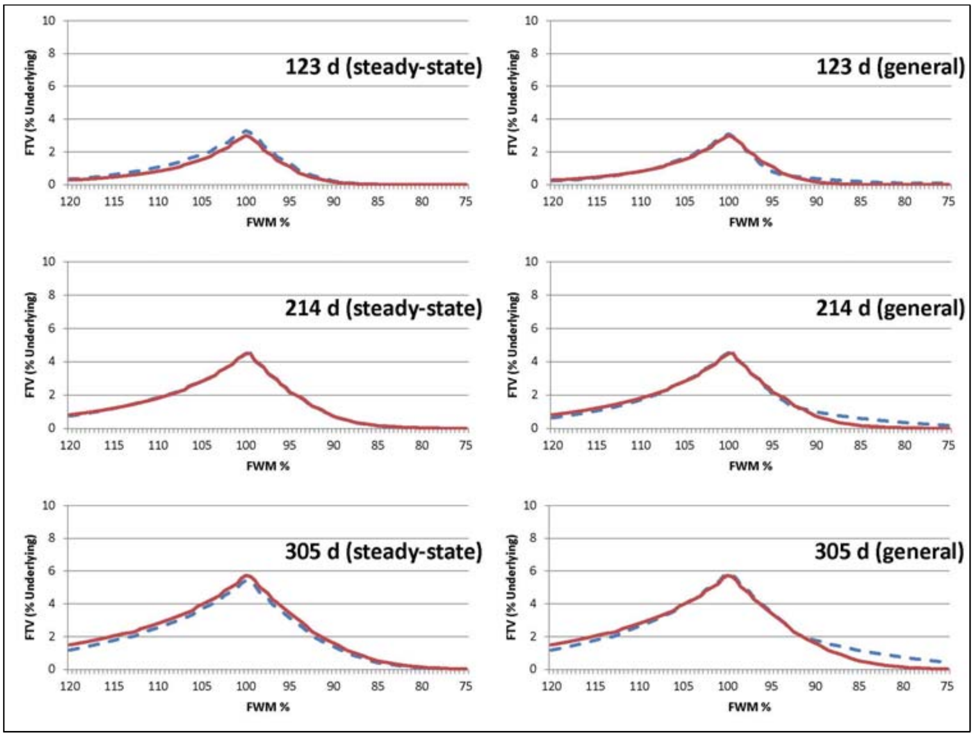


Comparison best steady-state and best general augmented model (part 1)

The prices of the 15th of May (figures on this page (short running options) and next page (long running options)) illustrate this phenomenon (solid lines = market prices). The following table shows the parameters of the two models and the norm distance to the market prices ($ND\ to\ m$):

	k_1	k_{2d}	$q_{1,2}$	$q_{2,1}$	q_1		$ND\,to\,m$
steady-state	20	215	0.1	0.02692	π_1		0.2110
gen. augm.	33	136	0.000401	0.00021	0		0.1490

$(\pi_1 = q_{2,1}/ (q_{1,2} + q_{2,1}) \approx 0,2121)$. Whereas in the steady-state model both regimes are entered and left again frequently, the best found general model starts in Z_2 und is likely to stay there for a long time. The expected value of the first time Z_1 is entered is 529 days!

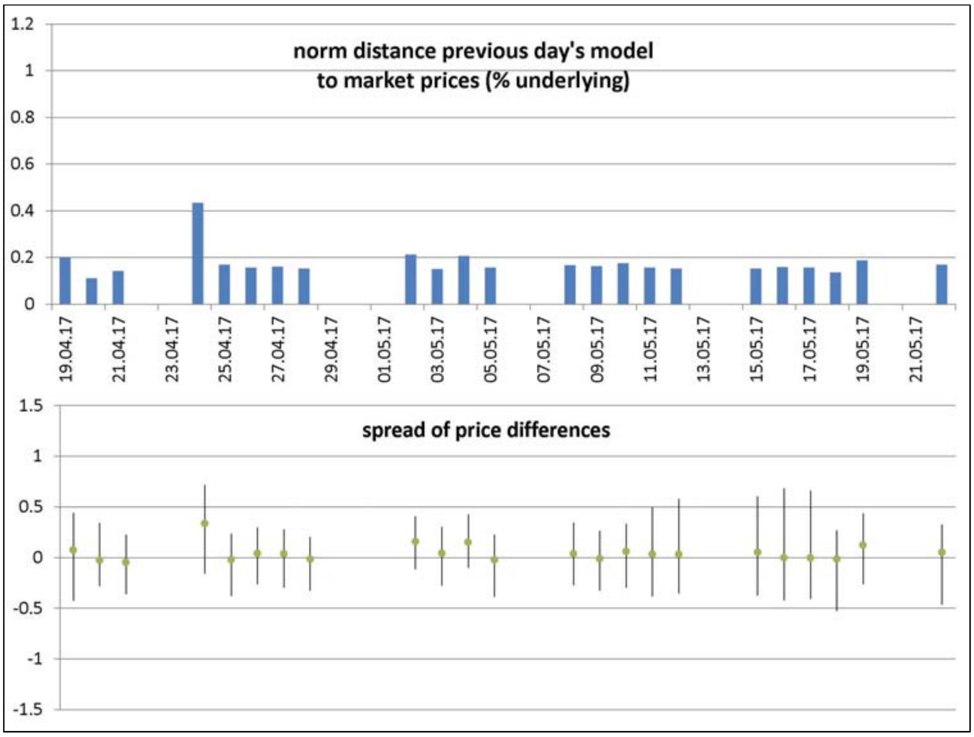


Comparison best steady-state and best general augmented model (part 2)

As a consequence, the kink is well visible in the forward time values of all maturities. Nevertheless, the improvement of ND is remarkable. Only for maturity 214 days is the contribution of the best steady-state model to ND better than that of the best general model. With 305 days maturity, however, the difference is small and the spread of the steady-state model is better (= smaller: $0.34 < 0.99$), whereas the average difference speaks in favour of the general augmented model (0.0038 vs 0.2557).

8.3 Stability

As in section 7.4 the overall best results in predicting tomorrow's option prices were obtained by the best model of today (reaugmented). The results of the individual trading days are shown in the following figure. The norm distance is very good throughout - with one exception. On Monday, the 24th of April, the day after the first ballot, the model of the preceding Friday generated option prices with a norm distance of 0.433 to the market prices - after reaugmentation.This can be a consequence of the fact that the selected model type does not provide for very large upward price jumps.



Quality of previous day's models

The second and third highest values are 0.212 (02.05.) and 0.205 (04.05.). All other values are less than 0.2. The average norm distance is 0.174. No forecasting strategy has been found that produces a better ND value than that. Although the best steady-state models have a higher stability in parameters and seem to be structurally more in accordance with market prices than the best general augmented models (see the example 15.05.), they only achieve an ND value of 0.197.

Also, no single set of parameters has been found (the "true model"), that produces a lower average norm distance when applied to all trading days. All investigated single sets of parameters have $ND > 0.24$. Even if only the days after the first ballot are included (24.04. - 22.05.), the best value is still > 0.185.

When comparing the French election period with the days of the Brexit vote, once again the French election period's 0.174 is better than the corresponding total average of the Brexit vote study (0.189). Removing the 24th of April, the "bad apple in the bunch", would even better the average to 0.163, a value that is close to, but just not as good as that of subperiod 1 or 3 of the Brexit vote study (cf page 68).

9 Summary

In the first part a model kit for option pricing models is introduced. Components are three types of regimes that can be represented by binary trees and are glued together by finite ergodic Markov chains. In these models, the process of the logarithm of the discounted value of a stock or a stock index under the price determining martingale measure is a stochastic process with stationary but not independent increments (steady-state models) or asymptotically stationary but not independent increments (general augmented models).

An important role is played by some features / modifications which are described by the names *lattice*, *variable time steps* and *tree-cut*. They allow to realize models with parameter combinations that would be out of reach otherwise on standard laptops. And they allow to treat maturities from 1 day to more than 1 year in one model. It is the empirical impression from the studies, that these modifications, in the way they are applied, do not distort relevant option prices. But, of course, an eye has to be kept on this aspect in the field.

Part 1 ends with a demonstration of several models that can be built with the model kit. From this it can be seen that quite a great variety of implied volatility surfaces (smiles and smirks) are in the scope of these models.

In Part 2 one of the simplest but nontrivial models, that can be built with the model kit, is tested on real option prices in times that were partially uneasy for political reasons. The models consist of two regimes - a BSM model and a skew ramification, the options are DAX® call options of Eurex. The periods investigated are three months containing the day of the Brexit vote in 2016 and about a month in 2017, in which the French presidential election took place.

Part 2 starts with a deeper introduction to the model type, which includes an illustration of the affect of the parameters on the option prices. Then the *norm distance* as measure between two sets of option prices is introduced (it is closely related to mean squares), and the optimization algorithm to find the best model-to-market fit is discussed. This is followed by the description of the results of the studies. It turns out that, with the exception of a few critical days, the option prices can be generated by steady-state models with good to very good accuracy. By general augmented models even the prices of the central critical days can be approximated at least to a good level.

The attempt to find a single model (i.e. one parameter set) that works well for all trading days of a complete period, was not completely successful. Better results in terms of norm distance are obtained if one always uses the best model of the preceeding trading day. This result supports a popular way of acting and maybe gets applauded by practitioners.

It may still be possible to find models that fit the market prices for a consecutive couple of days. But the studies show clearly that this will not be the case over a long period of time. Implied volatility surfaces from the Brexit period can easily be told from those of the days of the French election.

What remains to be done. First of all, certain clean-up operations must be carried out. For example, the relationship to continuous models must be fathomed more precisely and secured in such a way that, with sufficiently short time intervals, only the real parameters of the models are of essential importance and not the "secret ones" (see appendix).

Then, of course, systematic investigations on hedging are necessary in order to arrive at a sound assessment of the approaches in sections 7.4 and 8.3. In particular, a hedging strategy appropriate to the model must be developed, which means more than to solve the linear equations (9) on page 46.

A Appendix

A.1 Investigated Market Data: More Details

The data for this study were taken from the daily online market statistics of Eurex (daily settlement price of the options and closing price of the underlying), which are freely available for several weeks from the trading day on.

Maturities. The following maturities were included in the studies:

Brexit study (2016)							
02.05.-20.05.	Jun	Jul	Sep	Dec	Mar		
23.05.-31.05.	Jun	Jul	Aug	Sep	Dec	Mar	
. 01.06.-23.06.	Jun	Jul	Aug	Sep	Dec	Mar	Jun
24.06.-14.07.	Jul	Aug	Sep	Dec	Mar	Jun	
15.07.	Aug	Sep	Dec	Mar	Jun		
18.07.-29.07.	Aug	Sep	Oct	Dec	Mar	Jun	

French election (2017)						
18.04.-20.04.	Apr	May	Jun	Sep	Dec	Mar
21.04.	May	Jun	Sep	Dec	Mar	
24.04.-17.05.	May	Jun	Jul	Sep	Dec	Mar

French election (2017)	(cont.)					
18.05.	Jun	Jul	Sep	Dec	Mar	
22.05.	Jun	Jul	Aug	Sep	Dec	Mar

Non traded Options. Eurex publishes daily settlement prices not only for options that have been traded on that day, but also for many others (including restricted trading days on partial holidays as well as nationwide holidays with no trading at all (e.g. Whit Monday)). How these prices are computed cannot be seen in detail from the Eurex homepage. So the question was whether these prices should be used in the study or if investigations should better be restricted to traded options. Both ways were tested and the results were similar. But the impression was that the inclusion of options that were not traded on that day, led to somewhat smoother results. So the final decision was to use both, traded and not traded options (with or without open interest). This is all the more justified, as besides the norm distance some characteristic numbers like minimal / maximal difference between market price and model price were noted and can be found in chapters 7 and 8, that can only be better on a subset of the set of all prices than on the whole set.

Forward Moneyness. Call and put options only have a noteworthy (forward) time value if their forward moneyness is not too far away from 1. This is notably true for options with little time left to maturity. So in order not to give too much influence to options with almost zero time value, only options have been investigated with forward time value between 0.75 and 1.25. This is a compromise, as options with only a few days to maturity still have almost zero time value and options with a year or more time-to-maturity can have notable time value even if their forward moneyness is only 0.7.

Number of Option Prices. The approach explained above led to at least 370 and at most 555 option prices being included in the market data of each trading day. On average more than 430 prices were used. The number of maturities always was at least 5 and at most 7. The tables on page 81 show exactly what maturities were included.

Interest rates. The study was undertaken in times of very low and even negative interest rates. Whenever interest rates were needed in computations for any time period, the EONIA (Euro OverNight Index Average) of the first day of that period was taken. This was done in the strong belief that the precise value of interest rates close to zero does not have great influence on the prices of call options with not more than a year time to maturity.

During the whole periods, EONIA always was close to -0.3% p.a.

Time periods. The length of a time period was always taken to be the calendar length. That means, 1 day is 1/365 of a year (thereby ignoring that

2016 was a leap year). Financial usages like *1 year = 250 trading days* played no role.

A.2 Further Settings: Secret Parameters

In addition to the "official" parameters such as the upward and downward factors of the regimes, as well as the transition matrix and the initial distribution of the regimes, further specifications are required in order to set up a valuation tree. These values - also parameters in the strict sense - are required to have as little influence as possible on the option prices, ideally none at all. However, they are very important because they determine the calculability of a tree. The parameters used obviously led to calculable trees and have changed the option prices only reasonably. At least that's the impression, it is not formally proven.

Lattice parameters. See equations (7) in part I.

Variable time steps. These varied a little, but in the studies of the 2nd part the following basic pattern was used throughout:

k	1	2	3	4	5	6	7	8	9	10
$frequency$	37	15	13	10	8	10	10	10	10	10

(this means that the first 37 time intervals have length Δ_0, the next 15 have length $2^2\Delta_0$ etc.)

However, slight changes were necessary throughout as the maturities had to be met exactly.

Tree-cut. When building the tree, nodes with a probability of less than 10^{-8} were in danger of being cut off. But that was only a necessary criterion.

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